equations with respect to parabolic and hyperbolic equations. Thus, by studying the three equations in (1.6.1), we discover the properties of solutions to all equations of the form (1.6.2).

Exercises: Classification

1. Determine the set of points (x, y) where each of the following equations is elliptic, parabolic, and hyperbolic:
   (a) $x^2u_{xx}(x, y) - y^2u_{yy}(x, y) = 0$
   (b) $xyu_{xy}(x, y) - u_y(x, y) + u = 0$
   (c) $2u_{xx}(x, y) - 4u_{xy}(x, y) + u_x(x, y) = 0$
   (d) $2yu_{xx}(x, y) + (x + y)u_{xy}(x, y) + 2xu_{yy}(x, y) = 0$
   (e) $\sin(xy)u_{xy}(x, y) = 0$
2. Consider the equation

$$F(\mu) = \mu^2 - (a + c)\mu - (b^2 - ac) = 0$$

   Show that the following hold:
   (a) The graph of $F(\mu) = 0$ is a parabola opening upward.
   (b) If $b^2 - ac < 0$, then $F(\mu) = 0$ has two real roots of the same sign. In this case, if $(a + c) > 0$, then the roots are both negative, and if $(a + c) < 0$, they are both positive.
   (c) If $b^2 - ac = 0$, then $\mu = 0$ is a root of $F(\mu) = 0$. In this case the other root is positive or negative according to whether $a + c$ is negative or positive.
   (d) If $b^2 - ac > 0$, then $F(\mu) = 0$ has distinct roots of opposite sign.
3. Which of the following differential operators is linear?
   (a) $L[u] = u_{xx}(x, y) + x^2u_x(x, y) - u_{yy}(x, y)$
   (b) $L[u] = u_{xx}(x, y) - u(x, y)u_x(x, y) + u_{yy}(x, y)$
   (c) $L[u] = u_y(x, y)u_x(x, y) - u_{yy}(x, y)$
   (d) $L[u] = \sin(xy)u_{xx}(x, y) - \cos(xy)u_{xy}(x, y)$

Chapter 2

Fourier Series and Eigenfunction Expansions

INTRODUCTION

In this chapter we introduce the first of the two principal tools we shall be using to solve problems in partial differential equations: Fourier series, or more generally, eigenfunction expansions.

It is easy to check that for $n = 1, 2, \ldots, N$ and $\mu_n = n\pi/L$, the function

$$u_n(x, t) = \exp[-\mu_n^2 t]\sin(\mu_n x)$$

satisfies

$$u_t(x, t) = u_{xx}(x, t), \quad 0 < x < L, t > 0$$
$$u(0, t) = u(L, t) = 0$$

It is easily checked that for arbitrary constants $C_1, \ldots, C_N$ the function $U(x, t)$ given by

$$U(x, t) = \sum_{n=1}^{N} C_n u_n(x, t)$$

satisfies the partial differential equation and boundary conditions above. Clearly, at $t = 0$ $U(x, t)$ reduces to

$$U(x, 0) = \sum_{n=1}^{N} C_n \sin(\mu_n x)$$

and hence in order to satisfy an initial condition of the form

$$U(x, 0) = F(x), \quad 0 < x < L$$

it is necessary to be able to find values for the constants C_n such that

$$\sum_{n=1}^{N} C_n \sin(\mu_n x) = F(x), \qquad 0 < x < L$$

If such constants can be found, then $U(x, t)$ is seen to be the solution to the initial-boundary-value problem consisting of the partial differential equation together with the inhomogeneous initial condition and the two homogeneous boundary conditions. We show in Chapter 6 that if this problem has *any* solution, then that is the *only* solution. Evidently, we will have constructed the unique solution to this initial-boundary-value problem once the appropriate constants C_n are found.

For an arbitrary function $F(x)$ it is not, in general, possible for any *finite* value of N to find constants C_n such that $U(x, 0) = F(x)$. However, when N is allowed to become infinite, then for a large class of functions $F(x)$, the constants C_n can be chosen so that the (now infinite) series not only converges but converges to $F(x)$.

Although other mathematicians, including Euler, were working in the area as early as 1750, this discovery is generally attributed to the previously mentioned Joseph Fourier, and series of the sort to be discussed in this chapter are called Fourier series. In a classic work published about 1820, *Theorie Analytique de la Chaleur*, Fourier presented the mathematical techniques on which this chapter and the next are based. Although his ideas were essentially correct, they were not rigorously set forth, and Fourier had considerable difficulty in convincing other mathematicians of his time that it is possible to represent an arbitrary function as a series of periodic functions. Part of the confusion was no doubt a result of confusion about the precise meaning of the term *function*. In Sections 2.1 and 2.2 we introduce the notion of a *square integrable function*, and we shall see that this interpretation of *function* is particularly suitable for discussing Fourier series. A more substantial discussion of the space of square integrable functions is the subject of Section 2.5.

In section 2.2 we shall see that a Fourier series in terms of the sine and cosine functions is only a special case of a more general series expansion. Section 2.3 explains how to find families of functions that are suitable for use in these *generalized Fourier expansions*. Section 2.4 is a digression into the subject of *discrete Fourier series*, which is of interest in connection with the second part of the book, where discrete solution methods are discussed. Finally, in Section 2.6 we briefly describe the use of Fourier expansions for functions of several variables.

2.1 FOURIER SERIES

Let $f(x)$ denote an arbitrary function of x defined on the interval $(-L, L)$ and write

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \tag{2.1.1}$$

A series of this form, where the constants a_n and b_n are yet to be determined, is called a trigonometric infinite series. The series may or may not converge, and for those values of x where it does converge, it may or may not converge to the value $f(x)$. Of course,

for the series to converge to $f(x)$, it is necessary that the constants a_n and b_n depend in some way on the function $f(x)$. It will be our first task to show that (2.1.5) below expresses the necessary dependence of the constants a_n and b_n on $f(x)$.

Elementary integration formulas show that for arbitrary integers m, n

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n > 0 \\ 2L, & m = n = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n > 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0, \quad \text{all integers } m, n \tag{2.1.2}$$

The set of relations (2.1.2) are referred to as *orthogonality relations*, and the family of functions

$$\{1, \cos \pi x/L, \cos 2\pi x/L, \ldots; \sin \pi x/L, \sin 2\pi x/L, \ldots\}$$

is called an *orthogonal family* on the interval $(-L, L)$.

Proceeding formally, if we multiply both sides of (2.1.1) by $\cos M\pi x/L$ for a fixed integer M and then integrate from $-L$ to L, (2.1.1) becomes

$$\left(f, \cos\frac{M\pi x}{L}\right) \sim \tfrac{1}{2}a_0\left(1, \cos\frac{M\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n\left(\cos\frac{n\pi x}{L}, \cos\frac{M\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\left(\sin\frac{n\pi x}{L}, \cos\frac{M\pi x}{L}\right) \tag{2.1.3}$$

Here we are using the notation

$$(f, g) = \int_{-L}^{L} f(x)g(x)\,dx \tag{2.1.4}$$

for arbitrary functions $f(x)$ and $g(x)$ defined on $(-L, L)$. It follows from the orthogonality relations (2.1.2) that for every $n \neq M$,

$$(\sin n\pi x/L, \cos M\pi x/L) = 0$$
$$(\cos n\pi x/L, \cos M\pi x/L) = 0$$

Note that for $n = 0$, $\cos n\pi x/L = 1$.

Then if (2.1.1) is to be an equality, it must be the case that for each integer $M = 0, 1, \ldots,$

$$(f, \cos M\pi x/L) = a_M L$$

That is,

$$a_M = 1/L(f, \cos M\pi x/L) \quad \text{for } M = 0, 1, \ldots$$

Similarly, multiplying on both sides of (2.1.1) by $\sin M\pi x/L$ and integrating from $-L$ to L leads to

$$\left(f, \sin\frac{M\pi x}{L}\right) \sim \tfrac{1}{2}a_0\left(1, \sin\frac{M\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n\left(\cos\frac{n\pi x}{L}, \sin\frac{M\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\left(\sin\frac{n\pi x}{L}, \sin\frac{M\pi x}{L}\right).$$

Then the orthogonality relations (2.1.2) imply that in order for (2.1.1) to be an equation, we must have

$$(f, \sin M\pi x/L) = b_M L \quad \text{for each } M > 0$$

Evidently, a necessary condition for (2.1.1) to be an equality is that the coefficients a_n and b_n in the series are related to $f(x)$ as follows:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots \tag{2.1.5}$$

When the coefficients a_n and b_n in (2.1.1) are given by (2.1.5), they are called the *Fourier coefficients* for the function $f(x)$ and the series (2.1.1) is called the *Fourier series* for the function $f(x)$.

Up to this point we have shown only that *if* the series (2.1.1) is convergent to $f(x)$, then the coefficients a_n and b_n in the series *must* be given by (2.1.5). Of course, it was to be expected that the coefficients would depend on $f(x)$; (2.1.5) shows explicitly *how* a_n and b_n are determined from $f(x)$. Our next task will be to find conditions sufficient to ensure convergence of the series (2.1.1) with coefficients a_n and b_n given by (2.1.5). This development may be more meaningful if we first compute the coefficients a_n and b_n in a few simple examples and examine the resulting Fourier series. For convenience in these examples, we take $L = \pi$.

EXAMPLE 2.1.1

Consider

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases} \tag{2.1.6}$$

Note that *no value* has been specified for $f(x)$ at the three points $x = -\pi, 0, \pi$. Since the Fourier coefficients are calculated by integrating $f(x)$, this will have no effect on the Fourier coefficients (changing the value of the integrand at finitely many points does not affect the value of an integral), but it suggests an interesting point: Functions that differ at finitely many points have the same Fourier coefficients. We return to this point later.

For $n = 0, 1, \ldots$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx$$

After integrating, this reduces to

$$a_0 = 1$$
$$a_n = 1/(n\pi)\sin(nx)\big|_0^{\pi} = 0 \quad \text{for } n = 1, 2, \ldots$$

Note that the integration formula used to compute a_n in the case $n = 1, 2, \ldots$ was not the same one used to compute a_0.

Similarly,

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx \quad \text{for } n = 1, 2, \ldots$$
$$= \frac{-1}{n\pi}\cos(nx)\Big|_0^{\pi} = \frac{1-\cos(n\pi)}{n\pi}$$

Note that $\cos(n\pi) = (-1)^n$ for $n =$ integer. Then

$$b_n = \begin{cases} 0 & \text{if } n = \text{even integer} \\ 2/n\pi & \text{if } n = \text{odd integer} \end{cases}$$

This produces the following Fourier series for $f(x)$:

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$
$$\sim \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)x]}{2m-1} \tag{2.1.7}$$

If we define the partial sums of this series by

$$S_0(x) = \tfrac{1}{2}$$
$$S_1(x) = \tfrac{1}{2} + (2/\pi)\sin x$$
$$S_2(x) = \tfrac{1}{2} + (2/\pi)\sin x + (2/3\pi)\sin 3x$$
$$\vdots$$

then plotting $S_0, S_1, S_2 \ldots$ versus x shows that with each additional term, the graphs of the partial sums of this Fourier series draw closer to the graph of $f(x)$ (Figure 2.1.1). Note in particular that each of these three partial sums assumes the value $\frac{1}{2}$ at each of the points $x = -\pi, 0, \pi$ where $f(x)$ was left undefined. ■■

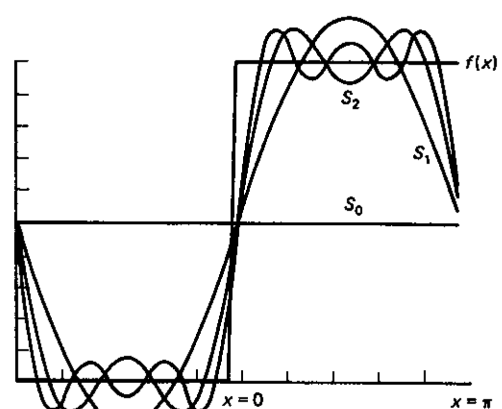


Figure 2.1.1

EXAMPLE 2.1.2

Consider the function

$$f(x) = x, \qquad -\pi < x < \pi$$

Using (2.1.2), we compute, for $n = 1, 2, \ldots,$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos(nx)\, dx = \frac{1}{\pi}\left[\frac{x \sin nx}{n} - \frac{(-\cos nx)}{n^2}\right]\Bigg|_{-\pi}^{\pi}$$
$$= 0$$

and for $n = 0$

$$a_0 = \int_{-\pi}^{\pi} x\, dx = 0$$

Similarly, for $n = 1, 2, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{1}{\pi}\left[\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2}\right]\Bigg|_{-\pi}^{\pi}$$
$$= \frac{-2\cos(n\pi)}{n} = \frac{2(-1)^{n+1}}{n}$$

Then the Fourier series for $f(x)$ in this example is

$$f(x) \sim 2 \sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x \ldots$$
$$\sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \qquad (2.1.8)$$

If we define

$$S_1(x) = 2 \sin x$$
$$S_2(x) = 2 \sin x - \sin 2x$$
$$S_3(x) = 2 \sin x - \sin 2x + \tfrac{2}{3}\sin 3x$$
$$\vdots$$

then plotting $S_1, S_2, S_3 \ldots$ versus x shows that including additional terms of the Fourier series improves the approximation to the function $f(x)$ (Figure 2.1.2). Note also that each of the partial sums $S_k(x)$ assumes the value zero at $x = -\pi, \pi$. ■■

EXAMPLE 2.1.3

Consider the function

$$f(x) = |x|, \qquad -\pi < x < \pi$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -x\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx = \pi$$

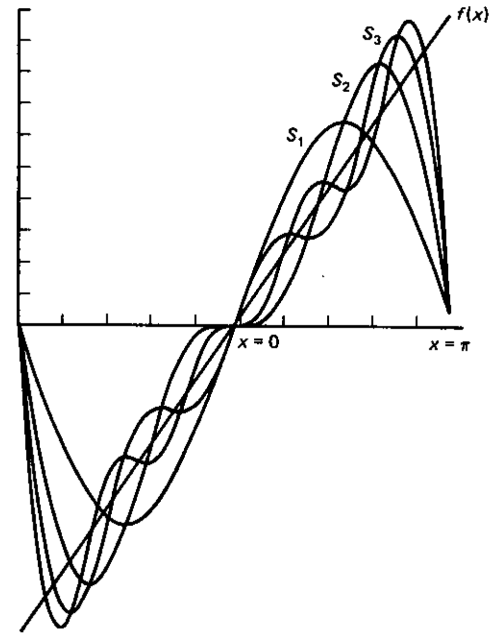


Figure 2.1.2

and for $n = 1, 2, \ldots$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x \cos nx\, dx$$
$$= -\frac{1}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]\Bigg|_{-\pi}^{0}$$
$$+ \frac{1}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]\Bigg|_{0}^{\pi}$$
$$= \frac{2}{\pi}\frac{(\cos n\pi - 1)}{n^2}$$

We also have, for $n = 1, 2, \ldots,$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -x\sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx\, dx$$
$$= \frac{1}{\pi}\left[\frac{x \cos nx}{n} - \frac{\sin nx}{n^2}\right]\Bigg|_{-\pi}^{0}$$
$$+ \frac{1}{\pi}\left[\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2}\right]\Bigg|_{0}^{\pi} = 0$$

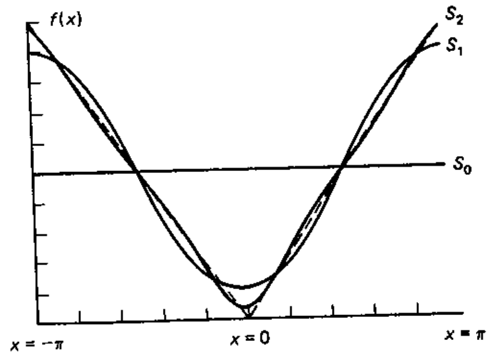


Figure 2.1.3

Then the Fourier series for $f(x)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \cdots$$
$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}[2m - 1]^{-2}\cos(2m - 1)x \qquad (2.1.9)$$

In this case, the partial sums of the Fourier series,

$$S_0(x) = \pi/2$$
$$S_1(x) = \pi/2 - (4/\pi)\cos x$$
$$S_2(x) = \pi/2 - (4/\pi)\cos x - (4/9\pi)\cos 3x$$

approach $f(x)$ "uniformly." That is, they have the property that for $k = 1, 2, \ldots$ the maximum difference $|S_k(x) - f(x)|$ for x between $-\pi$ and π is a decreasing function of k (see Figure 2.1.3) ■■

We now begin to develop the notions that will allow us to precisely describe and understand some of the things we have noticed about these examples.

Periodic Functions and Periodic Extensions

A function $f(x)$ defined for all values of x is said to be periodic with period P if

$$f(x) = f(x + P) \quad \text{for all } x \qquad (2.1.10)$$

For example:

sin x and cos x are periodic with period 2π.

sin nx and cos nx are periodic with period 2π for all integers n.

$\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ are periodic with period $2L$ for all integers n.

Any constant function is periodic with period P for every value P.

Note that if $f(x)$ is periodic with period P, then

$$f(x) = f(x + P) = f(x + 2P) = \cdots \quad \text{for all } x$$

Evidently, if $f(x)$ is periodic with period P, then $f(x)$ is also periodic with period Q for Q equal to any integer multiple of P. If it is necessary to speak of "the period" of $f(x)$ unambiguously, then the period of $f(x)$ is defined to be the *smallest* positive number P such that (2.1.10) holds.

For example, we have said that for each integer n the function sin nx is periodic with period 2π. This is so, but the smallest number P such that (2.1.10) holds for $f(x) = \sin nx$ is $P = 2\pi/n$. Thus,

sin x is periodic with period 2π.

sin $2x$ is periodic with period π.

sin $3x$ is a periodic with period $2\pi/3$.

And so on.

However, it is still correct to say that each of these functions is periodic with period 2π since (2.1.10) holds for each of these functions with $P = 2\pi$.

If $f_1(x), f_2(x), \ldots, f_N(x)$ are all periodic functions having a common period P, then for any integer N, the function

$$F(x) = \sum_{n=1}^{N} f_n(x)$$

is also periodic with period P. In particular, each of the partial sums $S_N(x)$ associated with the Fourier series (2.1.7), (2.1.8), and (2.1.9) is periodic with period 2π. Clearly, this period is determined by the first nonconstant term in the Fourier series.

A function $f(x)$ defined only on $(-L, L)$ can be extended to $(-\infty, \infty)$ as a periodic function of period $2L$ in the following way. For each number z, $-\infty < z < \infty$, there is a unique integer M such that $-L \le z - 2ML \le L$. Then we define $\tilde{f}(x)$ by

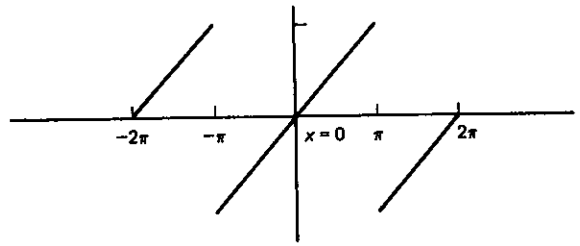
$$\tilde{f}(z) = f(z - 2ML) \quad \text{for } -\infty < z < \infty \qquad (2.1.11)$$

The function $\tilde{f}(x)$ is everywhere defined and is periodic with period $2L$, and $\tilde{f}(x)$ agrees with $f(x)$ on $(-L, L)$. The function $\tilde{f}(x)$ is called the *$2L$-periodic extension of $f(x)$*.

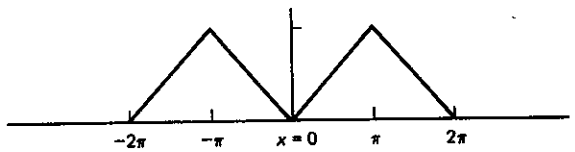
The functions of Examples 2.1.1, 2.1.2, and 2.1.3 are each defined on $(-\pi, \pi)$. The following sketches show the 2π-periodic extensions of each of these functions on the interval $(-2\pi, 2\pi)$:

Example 2.1.1:

Example 2.1.2:



Example 2.1.3:



Note the following facts about these extensions:

Example 2.1.1: the function $f(x)$ is not continuous on $(-\pi, \pi)$ and hence $\tilde{f}(x)$ is not continuous on $(-\infty, \infty)$.

Example 2.1.2: $f(x)$ is continuous on $(-\pi, \pi)$ but $\tilde{f}(x)$ is not continuous on $(-\infty, \infty)$.

Example 2.1.3: $f(x)$ is continuous and its extension is continuous. From these examples it is clear that the conditions under which the periodic extension is continuous are as follows:

Lemma 2.1.1. $\tilde{f}(x)$ is continuous on $(-\infty, \infty)$ if and only if

(a) $f(x)$ is continuous on $[-L, L]$ and
(b) $f(-L) = f(L)$.

There is something else we should notice about the examples. The Fourier series (2.1.8) contains only sine terms while the series (2.1.9) contains only cosine terms. To see why this is the case, note that for each integer n,

$\cos n\pi x/L$ satisfies

$$f(x) = f(-x) \quad \text{for all } x \tag{2.1.12}$$

$\sin n\pi x/L$ satisfies

$$f(x) = -f(-x) \quad \text{for all } x \tag{2.1.13}$$

Any function having property (2.1.12) is said to be an *even function of x* while any

function having property (2.1.13) is said to be an *odd function of x*. Most functions are neither even nor odd, but *any function* can be written as the sum of an even and an odd function as follows:

$$f(x) = f_E(x) + f_0(x) \tag{2.1.14}$$

where

$$f_E(x) = \tfrac{1}{2}[f(x) + f(-x)] = f_E(-x)$$

denotes the "even part of $f(x)$" and

$$f_0(x) = \tfrac{1}{2}[f(x) - f(-x)] = -f_0(-x)$$

denotes the "odd part of $f(x)$."

If

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}$$

then since $\cos n\pi x/L$ is *even* for $n = 0, 1, \ldots$ and $\sin n\pi x/L$ is *odd* for $n = 1, 2, \ldots$, it follows from (2.1.14) that

$$f_E(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$$

$$f_0(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$$

If $f(x)$ is known to be an even function on $(-L, L)$, then $f_0(x)$, the odd part of $f(x)$, is identically zero and hence $b_n = 0$ for every n. This is the case in Example 2.1.3. On the other hand, if $f(x)$ is known to be an odd function on $(-L, L)$, as in Example 2.1.2, then the even part of $f(x)$, $f_E(x)$, is identically zero and a_n vanishes for $n = 0, 1, \ldots$. When $f(x)$ is neither even nor odd, as in Example 2.1.1, the Fourier series for $f(x)$ will have both sine and cosine terms in it.

For every *even* function, not only can we anticipate that b_n is zero for $n = 1, 2, \ldots$ (without having to calculate the b_n's) but in addition it is the case that

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, \ldots \tag{2.1.15}$$

Likewise, for every *odd* function, in addition to anticipating that $a_n = 0$ for $n = 0, 1, \ldots$, we have

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots \tag{2.1.16}$$

Moreover, formulas (2.1.15) and (2.1.16) suggest the following additional consequence of these observations. Suppose we are given a function $f(x)$ defined only on the "half-interval" $(0, L)$. We have then two alternatives:

(a) We can *define* $b_n = 0$ for $n = 1, 2, \ldots$ and compute the Fourier coefficients a_n for $f(x)$ from (2.1.15). Then the Fourier series for $f(x)$ will have only cosine terms in it and must therefore represent an even function. We refer to this as the *half-range Fourier cosine series for* $f(x)$.

(b) We can *define* $a_n = 0$ for $n = 0, 1, 2, \ldots$ and compute the Fourier coefficients b_n for $f(x)$ from (2.1.16). In this case the Fourier series for $f(x)$ will have only sine terms in it and must then represent an odd function. We refer to this as the *half-range Fourier sine series for* $f(x)$.

Alternative (a) is equivalent to extending the function $f(x)$ to the interval $(-L, 0)$ as an *even function* and then computing the Fourier coefficients in the usual way from (2.1.6). Alternative (b) is equivalent to extending $f(x)$ to the interval $(-L, 0)$ as an *odd function*. We refer to these extensions as the even and odd extensions of $f(x)$. If these extensions are themselves extended to the whole real line as $2L$-periodic functions, we obtain what we call the even $2L$-periodic and odd $2L$-periodic extensions for $f(x)$. In the next section we establish the fact that the half-range Fourier cosine–sine series for a function converge respectively to the even–odd $2L$-periodic extension of that function. In the next chapter we shall see some applications of half-range series.

We have observed that if $f(x)$ defined on $(-L, L)$ is symmetric about the y axis, then the Fourier series for $f(x)$ contains no sine terms, and if $f(x)$ is symmetric about the origin, then the Fourier series contains no cosine terms. Other types of symmetry cause other terms to be absent from the Fourier series for $f(x)$. We shall not pursue this here, but some of the problems at the end of this section explore the connection between symmetries and the Fourier coefficients.

One point we do wish to make note of, however, is that the initial term in any Fourier series is the term

$$\tfrac{1}{2}a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Clearly this is just the average value of the function $f(x)$ on the interval $(-L, L)$. Then we can think of the Fourier series for $f(x)$ as composed of the (constant) average-value component plus all the periodic components that oscillate about this average value with amplitudes equal to the Fourier coefficients.

Convergence of Fourier Series

We begin our discussion of convergence for Fourier series by recalling a few facts about convergence of infinite series of functions. Consider then an infinite family of functions $u_1(x), u_2(x), \ldots$, all defined on a common interval I. Associated with this family of functions consider the infinite series

$$\sum_{k=1}^{\infty} u_k(x) \tag{2.1.17}$$

If each of the functions $u_k(x)$ is of the form $u_k(x) = a_k(x - x_0)^k$, then the infinite series is called a *power series*. If the terms $u_k(x)$ are all trigonometric functions, then the series is called a *trigonometric series*. Fourier series are a special type of trigonometric series.

The series (2.1.17) defines a function $S(x)$ that we refer to as the "sum" of the series. The domain of $S(x)$ is the set of points x in I where the series converges and the value of $S(x)$ at a point of convergence is the value to which the series converges at the point. Of course, this set of points may be empty, but as we shall see, under certain conditions on the functions $u_k(x)$ in (2.1.17), the domain of $S(x)$ is not empty.

For $N = 1, 2, \ldots$ let $S_N(x)$ be defined by

$$S_N(x) = \sum_{k=1}^{N} u_k(x) \quad \text{for } x \text{ in } I$$

Then the functions $S_1(x), S_2(x), \ldots$ form what is called the *sequence of partial sums* for the infinite series (2.1.17). The convergence of the infinite series can now be defined in terms of the convergence of the sequence of partial sums. In fact, there is more than one "mode" of convergence we need to define.

Definition. The infinite series (2.1.17) is said to converge to the sum $S(x)$ on the interval I,

(a) in the *mean-square sense* if

$$\int_I |S_N(x) - S(x)|^2\,dx \to 0 \quad \text{as } N \to \infty$$

(b) in the *pointwise sense* if

$$\text{For each } x \text{ in } I, \quad |S_N(x) - S(x)| \to 0 \quad \text{as } N \to \infty$$

(c) *uniformly on I* if

$$\max_I |S_N(x) - S(x)| \to 0 \quad \text{as } N \to \infty$$

Each of these modes of convergence provides a way of describing how the graph of $S_N(x)$ draws close to that of $S(x)$ as N tends to infinity. For example, when the series (2.1.17) converges *uniformly* on I to $S(x)$, then for each $\epsilon > 0$ there is an integer $N_\epsilon > 0$ and an "ϵ-strip,"

$$E_\epsilon = \{(x, y) : S(x) - \epsilon < y < S(x) + \epsilon;\ x \text{ in } I\}$$

such that for every $N > N_\epsilon$ the graph of $S_N(x)$ is entirely contained in E_ϵ. This is the case, for example, in Figure 2.1.3.

If the series (2.1.17) converges pointwise to $S(x)$, this means only that for each x in I, $S_N(x)$ converges to $S(x)$ but the *rate* of convergence may vary with x. In Figure 2.1.2 we see an example where $S_N(x)$ appears to be converging to $S(x)$ fairly rapidly for x between -0.9π and 0.9π, but the convergence for $x = 0.99\pi$ is evidently slower. This is an example of pointwise convergence that is not uniform.

Finally, mean-square convergence of $S_N(x)$ to $S(x)$ ensures only that the area contained between the graphs of $S_N(x)$ and $S(x)$ decreases to zero as N tends to infinity. This does not then necessarily mean that $S_N(x)$ tends to the value $S(x)$ at every x in I. In many practical situations where our knowledge of the functions involved is not sufficiently precise to support pointwise evaluations, it is natural to think of the mean-square mode of convergence.

Every series that is uniformly convergent is also convergent in the mean-square and pointwise senses. However, there are mean-square convergent series that converge neither pointwise nor uniformly, and there are pointwise convergent series that do not converge in the mean-square sense and do not converge uniformly. A "litmus test" for distinguishing uniform from nonuniform convergence is contained in the following.

Proposition 2.1.1. Suppose the series (2.1.17) converges uniformly on I to the sum $S(x)$. If each of the functions $u_n(x)$ is continuous on I, then $S(x)$ is necessarily continuous on I.

It follows from this proposition that the series in Examples 2.1.1 and 2.1.2 cannot converge uniformly on any interval I of length longer than 2π since any such interval will contain a discontinuity of the limit function $S(x)$.

A useful test for determining if a given series does converge uniformly is the so-called Weierstrass M-test. A more complete discussion of this and the previous result can be found in most advanced calculus texts. A very good nontechnical discussion can be found in *Foundations of Applied Mathematics* by Michael D. Greenberg (Prentice-Hall, 1978).

Proposition 2.1.2 (Weierstrass M-test). A sufficient condition for the uniform convergence on I of the series (2.1.17) is the existence of a convergent series of positive constants ΣM_n such that for all x in I, $|u_n(x)| \le M_n$ for $n = 1, 2, \ldots$.

Note that according to this proposition, the series in (2.1.9) is uniformly convergent on any interval I of finite length.

Consider now the following special case of the series (2.1.17),

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \tag{2.1.18}$$

Whether or not this is a Fourier series for some function $f(x)$, we can use tests such as the Weierstrass M-test to determine whether the series converges in one mode or another. Conditions sufficient to ensure convergence of the series must then bear on the coefficients a_n and b_n and can imply nothing about the sum $S(x)$ to which the series converges. When (2.1.18) *is* a Fourier series for a given $f(x)$, then it is our aim to determine conditions which are sufficient to ensure that the Fourier series *converges to* $f(x)$. It is clear that such conditions must then bear on the function $f(x)$.

In order to state these conditions on $f(x)$, it will be convenient to define the notion of a sectionally continuous function. First we introduce the notation $f(x^+)$ and $f(x^-)$ to denote the limit of $f(x)$ as we approach x from the right and left, respectively. Note that any point x where $f(x^+) = f(x^-)$ is a point of continuity for $f(x)$. Then:

Definition. $f(x)$ is said to be *sectionally continuous* if

(a) at each point x the limits $f(x^+)$ and $f(x^-)$ both exist and are finite and
(b) in any interval of finite length there are at most finitely many x such that $f(x^+)$ is not equal to $f(x^-)$.

Obviously every continuous function is sectionally continuous. The functions in Examples 2.1.1 and 2.1.2 are examples of functions that are sectionally continuous but not continuous. The function $f(x) = x^{-1}$ is an example of a function that is not sectionally continuous on any interval that contains $x = 0$ since neither $f(0^+)$ nor $f(0^-)$ are finite in this case.

The class of sectionally continuous functions is a class of functions that is strictly larger than the class of continuous functions. A still larger class of functions is the class of square-integrable functions. The function $f(x)$ is said to be *square integrable* on $(-L, L)$ if

$$\int_{-L}^{L} f(x)^2\, dx < \infty$$

Every sectionally continuous function is square integrable on $(-L, L)$, but the converse is false. For example, $f(x) = x^{-1/4}$ is square integrable but is not sectionally continuous on $(-L, L)$. We shall have more to say about square-integrable functions later in the chapter.

Now we can state some conditions under which the Fourier series for $f(x)$ converges to $f(x)$.

Theorem 2.1.1. Suppose that $f(x)$ is defined on the interval $(-L, L)$ and let $\tilde{f}(x)$ denote the $2L$-periodic extension of $f(x)$.

(a) If $f(x)$ is square integrable on $(-L, L)$, then the Fourier series for f converges to $f(x)$ in the mean-square sense on $(-L, L)$.
(b) If $\tilde{f}(x)$ and its derivative $d/dx[\tilde{f}(x)]$ are both sectionally continuous, then at each x the Fourier series for $f(x)$ converges pointwise to the value

$$\tfrac{1}{2}[\tilde{f}(x^+) + \tilde{f}(x^-)]$$

(c) If $\tilde{f}(x)$ is continuous and $[\tilde{f}(x)]'$ is sectionally continuous, then the Fourier series for $f(x)$ converges uniformly to $\tilde{f}(x)$.

The theorem has the following corollary describing the conditions under which we can differentiate the Fourier series for $f(x)$ and expect the differentiated series to converge to the derivative of $f(x)$.

Corollary. With the notation of the theorem we have:

(d) If $\tilde{f}(x)$ is continuous and if $[\tilde{f}(x)]'$ and $[\tilde{f}(x)]''$ are both sectionally continuous, then the Fourier series for $f(x)$ converges uniformly to $\tilde{f}(x)$. In addition, the Fourier series for $f(x)$ may be differentiated, term by term, and at each x the differentiated series converges pointwise to the value

$$\tfrac{1}{2}[\tilde{f}(x^+)]' + \tfrac{1}{2}[\tilde{f}(x^-)]'$$

(e) If $\tilde{f}(x)$ and $[\tilde{f}(x)]'$ are both continuous and if the second derivative, $[\tilde{f}(x)]''$, is sectionally continuous, then the Fourier series for $f(x)$ converges uniformly to $\tilde{f}(x)$. In addition, the Fourier series for $f(x)$ may be differentiated, term by term, and this differentiated series converges uniformly to $[\tilde{f}(x)]'$.

Note that with the exception of part (a) of the theorem, the convergence properties of the Fourier series for $f(x)$ are determined by the smoothness properties of the periodic extension $\tilde{f}(x)$; the smoother the function $\tilde{f}(x)$, the stronger is the sense in which the Fourier series for $f(x)$ converges. Observe also that we do not mention differentiating the Fourier series for $f(x)$ unless $\tilde{f}(x)$ is at least continuous with a sectionally continuous derivative. Without at least this much smoothness the derivative of $\tilde{f}(x)$ may fail to exist at some points and the differentiated Fourier series could not converge.

Proving the Fourier convergence theorem is a difficult exercise in real analyis. Such an effort would be out of place in this book, whose focus is intended to be the solution of partial differential equations. Instead we illustrate the meaning and use of the theorem with some examples.

EXAMPLE 2.1.4

Consider the function

$$g(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$$

Then $g(x)$ is related to the function $f(x)$ of Example 2.1.1 by the equation

$$g(x) = 2[f(x) - \tfrac{1}{2}]$$

and the Fourier coefficients for $g(x)$ are related to the Fourier coefficients for $f(x)$ in the same way. That is,

$$g(x) = \frac{4}{\pi}\sum_{m=1}^{\infty} (2m-1)^{-1}\sin(2m-1)x$$

The 2π-periodic extension of $g(x)$ is the function $\tilde{g}(x)$ whose graph on the interval $[-3\pi, 3\pi]$ is shown in Figure 2.1.4. It is evident from the graph that $\tilde{g}(x)$ is sectionally continuous but not continuous. In particular,

$$\tilde{g}(0^+) = +1 \quad \text{does not equal} \quad -1 = \tilde{g}(0^-)$$
$$\tilde{g}(\pi^+) = -1 \quad \text{does not equal} \quad +1 = \tilde{g}(\pi^-), \quad \text{etc.}$$

Thus $\tilde{g}(x)$ has a "jump discontinuity" at every integer multiple of π, but on any interval of finite length, there are only finitely many of these jumps. The derivative $[\tilde{g}(x)]'$ is also sectionally continuous; in fact, the derivative of $\tilde{g}(x)$ is zero at each point where it is defined. At the integer multiples of π, the derivative is not defined, but the left and right limiting values for $[\tilde{g}(x)]'$ at these points do exist.

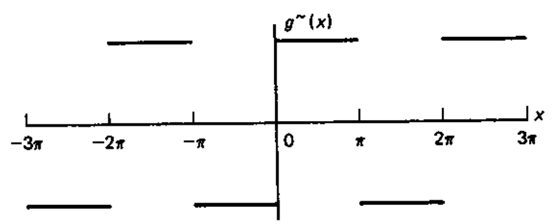


Figure 2.1.4

It follows then from part (b) of the Fourier convergence theorem that the Fourier series for $g(x)$ converges pointwise to the value

$$\tfrac{1}{2}[\tilde{g}(x^+) + \tilde{g}(x^-)] = \begin{cases} \tilde{g}(x) & \text{at each } x \text{ where } \tilde{g} \text{ is continuous} \\ 0 & \text{at integer multiples of } \pi \end{cases}$$

Note that since $\tilde{g}(x)$ is not continuous, the convergence of this Fourier series could not be uniform as this would violate Proposition 2.1.1 ■■

Note also that the differentiated Fourier series does not converge since the nth term of the differentiated series fails to go to zero. This is consistent with the fact that $\tilde{g}(x)$ is not continuous and hence lacks the minimal amount of smoothness required for differentiating its Fourier series.

EXAMPLE 2.1.5

Consider the function

$$G(x) = |x|, \qquad -\pi < x < \pi$$

This is just the function of Example 2.1.3, where we found the Fourier series to be

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} (2m-1)^{-2}\cos(2m-1)x$$

The graph of Example 2.1.3 of the periodic extension $\tilde{G}(x)$ shows that $\tilde{G}(x)$ is continuous for all values of x. The derivative $[\tilde{G}(x)]'$ is just the function $\tilde{g}(x)$ from Example 2.1.4 and hence $[\tilde{G}(x)]'$ is sectionally continuous. Then part (c) of the Fourier convergence theorem implies that the Fourier series for $\tilde{G}(x)$ converges uniformly to $\tilde{G}(x)$ on $(-\infty, \infty)$.

Compare this with the function

$$F(x) = x, \qquad -\pi < x < \pi$$

whose Fourier series was found in Example 2.1.2 to be given by (2.1.8). The graph of the periodic extension $\tilde{F}(x)$ was also shown previously, and it is clear that $\tilde{F}(x)$ is sectionally continuous but not continuous. The derivative $[\tilde{F}(x)]'$ is also sectionally continuous; $[\tilde{F}(x)]' = 1$ at each point where it is defined. At odd-integer multiples of π, $\tilde{F}(x)$ has a finite jump discontinuity, and so the derivative $[\tilde{F}(x)]'$ is not defined, although its left and right limits at these points do exist. Then part (b) of the Fourier convergence theorem implies that the Fourier series for $F(x)$ converges pointwise to the value

$$\tfrac{1}{2}[\tilde{F}(x^+) + \tilde{F}(x^-)] = \begin{cases} \tilde{F}(x) & \text{at points of continuity} \\ 0 & \text{at odd-integer multiples of } \pi \end{cases}$$

It is striking that although both $F(x)$ and $G(x)$ are continuous on the interval $(-\pi, \pi)$ and, in fact, $F(x)$ is the smoother of the two functions there, the Fourier series for $G(x)$ converges more strongly than the Fourier series for $F(x)$. Evidently the convergence of the Fourier series for a function is controlled by the smoothness of the periodic extension of the function. As Lemma 2.1.1 implies, smoothness of the function does not necessarily imply smoothness of its periodic extension.

Note that the function $\tilde{G}(x)$ is continuous and $\tilde{G}(x)$ has as its derivative the sectionally continous function $\tilde{g}(x)$, which also has a sectionally continuous derivative [which is then the second derivative of $\tilde{G}(x)$]. Then part (d) of the corollary to Theorem 2.1.1 implies that the Fourier series for $G(x)$ may be differentiated, term by term, and the resulting series must then converge pointwise to the value

$$\tfrac{1}{2}[\tilde{G}(x^+)]' + \tfrac{1}{2}[\tilde{G}(x^-)]' = \tfrac{1}{2}[\tilde{g}(x^+) + \tilde{g}(x^-)]$$

This conclusion can be verified by differentiating the Fourier series for $G(x)$, term by term, and observing that one does in fact get the Fourier series for $g(x)$. On the other hand, $\tilde{F}(x)$ does not satisfy the conditions of the corollary to Theorem 2.1.1, and one sees that differentiating the Fourier series (2.1.8) for $F(x)$ leads to the series

$$2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$$

Not only does this series not converge to the derivative of $F(x)$, but the series is obviously divergent since the nth term does not go to zero. As with the function $\tilde{g}(x)$ in the previous example, $\tilde{F}(x)$ lacks the minimal smoothness required for differentiating the Fourier series. ■■

Exponential Form of Fourier Series

We have seen that under certain conditions a function $f(x)$ defined on the interval $(-L, L)$ can be represented as an infinite series of the form (2.1.1). In view of the Euler identity, $\exp[i\theta] = \cos\theta + i\sin\theta$ (i denotes $\sqrt{-1}$), the series has the alternative form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \exp\frac{in\pi x}{L} \qquad (2.1.19)$$

where the coefficients c_n can be expressed in terms of the a_n and b_n by substituting the Euler identity into (2.1.19), collecting terms, and comparing with (2.1.1).

Instead we will find the coefficients c_n directly in terms of $f(x)$ as follows. For n equal to any integer, let

$$E_n(x) = \exp[in\pi x/L] \qquad (2.1.20)$$

and introduce the notation

$$«F, G» = \int_{-L}^{L} F(x)\overline{G}(x)\,dx \qquad (2.1.21)$$

Here $\overline{G}(x)$ in the integrand denotes the complex conjugate of $G(x)$. In particular,

$$\overline{E}_n(x) = \exp[-in\pi x/L]$$

Then it is easily shown that

$$«E_m, E_n» = \begin{cases} 0, & m \neq n \\ 2L, & m = n \end{cases} \qquad (2.1.22)$$

These are the orthogonality relations for the family of functions $\{E_n(x) : n =$ integ the interval $(-L, L)$. We can use the relations (2.1.22) to find the coefficients c_n same way we used the orthogonality relations (2.1.2) to find the a_n and b_n. Tha (2.1.19) is an equation, then (formally at least) we have, for fixed integer M,

$$«f, E_M» = \sum_{-\infty}^{\infty} c_n «E_n, E_M» = c_M 2L$$

This leads to

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\overline{E}_n(x)\,dx \qquad (2$$

The series (2.1.19) with the coefficients c_n given by (2.1.23) is called the *expo form of the Fourier series*. The exponential form of the Fourier series is com equivalent to the form (2.1.1), and in fact, substituting the Euler identity into (2 shows immediately that

$$c_n = \begin{cases} \tfrac{1}{2}[a_n - ib_n] & \text{for } n = 0, 1, \ldots \\ \tfrac{1}{2}[a_n + ib_n] & \text{for } n = -1, -2, \ldots \end{cases} \qquad (2$$

In view of this equivalence, the Fourier convergence theorem applies v change to the exponential form of the series.

EXAMPLE 2.1.6

Consider the function defined on the interval $(-\pi, \pi)$ by

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } -\pi < x < -a \quad \text{or} \quad a < x < \pi \end{cases}$$

Here a denotes a positive constant less than π.

Then

$$c_n = \frac{1}{2\pi}\int_{-a}^{a} 1 \exp[-inx]\,dx$$
$$= \frac{1}{-2n\pi i}\exp[-inx]\Big|_{-a}^{a}$$
$$= \frac{1}{n\pi}\sin na, \qquad n = -1, 1, -2, 2, \ldots$$

and $c_0 = a/\pi$.

Then

$$f(x) = \frac{a}{\pi} + \sum_{n<0}\frac{1}{n\pi}(\sin na)e^{inx} + \sum_{n>0}\frac{1}{n\pi}(\sin na)e^{inx}$$

This is the exponential form of the Fourier series for this function $f(x)$. Note tha

$$\sin(na)/n = \sin(-na)/(-n)$$

and hence the sum over positive n and the sum over negative n can be combined to give

$$f(x) = \frac{a}{\pi} + \sum_{n>0} \frac{1}{n\pi} \sin na[e^{inx} + e^{-inx}]$$
$$= \frac{a}{\pi} + \sum_{n>0} \frac{2}{n\pi} \sin na \cos nx$$

That is, this is the trigonometric form of the Fourier series for this $f(x)$. This example clearly illustrates the equivalence of the exponential and trigonometric forms of the Fourier series. ■■

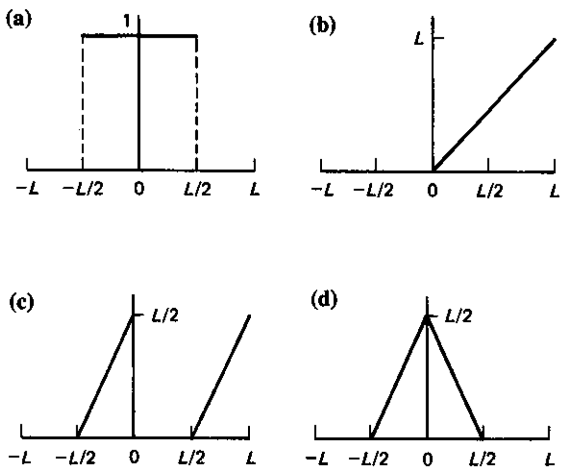
Our discussion of Fourier series up to this point has dealt with the so-called classical aspects of the subject. We have seen that when the coefficients a_n and b_n are related to the function $f(x)$ as described in (2.1.5), then the series (2.1.1) converges to the function $f(x)$. The mode of convergence is dependent on certain properties of $f(x)$ explained in Theorem 2.1.1. The conditions sufficient to ensure convergence for the Fourier series are mild compared to what is required to ensure convergence of other forms of infinite series. For example, power series converge to a function in a neighborhood of a point only if that function is analytic at the point.

We have seen also that the Fourier series may be expressed in the alternative form (2.1.19). Of course, this form of the series is completely equivalent to the form (2.1.1); in fact, the coefficients c_n and the coefficients a_n and b_n are related by (2.1.24). In addition, for functions $f(x)$ defined only on the "half-interval" $(0, L)$, we have the so-called half-range sine or cosine series, which converge to $f(x)$ on $(0, L)$ and converge, respectively, to the odd or even extension of $f(x)$ on the interval $(-L, 0)$.

All of this is part of a more general theory we develop in the next section.

Exercises: Fourier Series

The following functions are defined on $(-L, L)$.



1. Compute the Fourier coefficients a_n and b_n for each of these functions and sketch the graph on $(-2L, 2L)$ of the function to which the Fourier series converges.
2. Compute the coefficients c_n for the exponential form of the Fourier series for each of the functions (a)–(d).
3. Tell whether the Fourier series for the functions $f(x)$ in (a)–(d) converge pointwise or uniformly. Tell whether the differentiated series converges to the derivative of $f(x)$.
4. Compute the Fourier coefficients a_n and b_n for each of the following functions:
   (a) $f(x) = \sin x, \quad -\pi < x < \pi$
   (b) $f(x) = \sin \frac{1}{2}x, \quad -\pi < x < \pi$
   (c) $f(x) = |\sin x|, \quad -\pi < x < \pi$
   (d) $f(x) = \begin{cases} \cos x, & -\pi/2 < x < \pi/2 \\ 0, & -\pi < x < -\pi/2,\ \pi/2 < x < \pi \end{cases}$
   (e) $f(x) = \begin{cases} \sin x, & -\pi/2 < x < \pi/2 \\ 0, & -\pi < x < -\pi/2,\ \pi/2 < x < \pi \end{cases}$
5. In parts (a), (b), and (c) of Exercise 4, each of these Fourier series is missing some terms; that is, some of the coefficients a_n and b_n are zero. In each case, explain the reason for the missing terms. For example, the $f(x)$ in (c) is symmetric about the lines $x = 0$ and $x = \pi/2$. Such symmetry will cause some of the Fourier coefficients to vanish. In each of the five cases, calculate the coefficients, see which, if any, vanish, and examine the graph of $f(x)$ for symmetries. Then try to connect the missing terms with the symmetries.
6. Compute the Fourier coefficients a_n and b_n for the following functions:
   (a) $f(x) = x^2, \quad -1 < x < 1$
   (b) $f(x) = |x|, \quad -1 < x < 1$
   (c) $f(x) = \begin{cases} x + 1 & \text{if } -1 < x < 0 \\ 1 - x & \text{if } \ 0 < x < 1 \end{cases}$
   (d) $f(x) = 1 - x^2, \quad -1 < x < 1$
7. For each of the functions $f(x)$ in Exercise 6, sketch the graph on the interval $(-2, 2)$ of the periodic extension $\tilde{f}(x)$. Also sketch the graph of the derivative of the periodic extension. Does the differentiated Fourier series converge to this derivative in any of the four cases?
8. Write the Fourier series for the functions:
   (a) $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } -\pi < x < -\pi/2,\ \pi/2 < x < \pi \end{cases}$
   (b) $g(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } -\pi < x < -\pi/2,\ \pi/2 < x < \pi \end{cases}$
   (c) $h(x) = \begin{cases} 2x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } -\pi < x < -\pi/2,\ \pi/2 < x < \pi \end{cases}$
9. Use the Fourier series of Exercise 8 to discuss the following: Which of the functions $f(x)$ or $g(x)$ has as its derivative the function $h(x)$? Explain what is wrong with the following statement: "Since $f(x)$ and $g(x)$ differ by a constant, they have the same derivative." Each of the functions f and g is an antiderivative of h. Which of the two has as its Fourier series an antiderivative of the Fourier series for h?
10. Each of the following functions is defined on the interval $(0, 1)$.
   (a) $f(x) = \begin{cases} 0 & \text{if } a < x < 1,\ a = \text{positive const} \\ 1 & \text{if } 0 < x < a \end{cases}$

(b) $f(x) = \begin{cases} 1 & \text{if } \frac{1}{2} - s < x < \frac{1}{2} + s,\ s = \text{const},\ 0 < s < \frac{1}{2} \\ 0 & \text{if } 0 < x < \frac{1}{2} - s,\ s + \frac{1}{2} < x < 1 \end{cases}$

For each of the functions (a) and (b) write a Fourier cosine series. Sketch the graph on the interval $(-2, 2)$ to which the series converges. Does the cosine series converge pointwise or uniformly? Discuss whether the differentiated Fourier series converges to the derivative of $f(x)$.

11. For each of the functions (a) and (b) in Exercise 10 write a Fourier sine series. Sketch the graph on the interval $(-2, 2)$ to which the series converges. Does the sine series converge pointwise or uniformly? Discuss whether the differentiated Fourier series converges to the derivative of $f(x)$.

12. Use the Fourier half-range series to discuss the following statements:

(a) The derivative of an even/odd function is odd/even.
(b) If $f(x)$ is defined on $[-L, L]$ and is odd, then $f(x)$ vanishes at $x = 0, L, -L$.
(c) If $f(x)$ is defined on $[-L, L]$ and is even, then the derivative $f'(x)$ vanishes at the points $x = 0, L, -L$.

Tell whether each statement is true or false. If the statement is true only under special conditions on $f(x)$, describe those conditions.

13. The following functions are defined on the interval $(0, \pi)$:

$$f_0(x) = x, \qquad 0 < x < \pi$$

$$f_1(x) = \begin{cases} x & \text{if } 0 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < \pi \end{cases}$$

$$f_2(x) = \begin{cases} x & \text{if } 0 < x < \pi/2 \\ x - \pi & \text{if } \pi/2 < x < \pi \end{cases}$$

Write a Fourier sine series and Fourier cosine series for each of the three functions. Then answer the following questions:
(a) For each of the six series, sketch the graph on $(-\pi, \pi)$ of the function to which the series converges.
(b) For which function does the sine series contain no terms of the form (i) $\sin 2nx$ or (ii) $\sin(2n - 1)x$.
(c) For which function does the cosine series contain no terms of the form (i) $\cos 2nx$ or (ii) $\cos(2n - 1)x$.

14. In order that $f(x)$ defined on $[-\pi, \pi]$ has a continuous, 2π-periodic extension, it is necessary and sufficient that (i) $f(x)$ is continuous on $[-\pi, \pi]$ and (ii) $f(\pi) = f(-\pi)$. Under what conditions is the 2π-periodic extension continuously differentiable (i.e., $\tilde{f}$ is continuous, together with its first derivative)? Carry this further and find conditions on f sufficient to ensure that $\tilde{f}$ is continuous, together with each of its derivatives up to order M.

15. Suppose that $f(x)$ defined on $[-\pi, \pi]$ has a continuously differentiable 2π-periodic extension. Then show that the Fourier coefficients for $f(x)$ must satisfy

$$|a_n| \leq C/n \quad \text{and} \quad |b_n| \leq C/n, \qquad n = 1, 2, \ldots$$

where $C > 0$ denotes a constant that is independent of n. Show that if $\tilde{f}$ is continuous, together with its derivatives up to order M, then there exists a constant $C > 0$ such that

$$n^M |a_n| \leq C \quad \text{and} \quad n^M |b_n| \leq C, \qquad n = 1, 2, \ldots$$

[*Hint:* Integrate by parts.]

16. The function $f(x) = -x$ is defined for $0 < x < 1$. Show that the Fourier sine series for $f(x)$ is

$$f(x) = 2 \sum_{n=1}^{\infty} (n\pi)^{-1} \sin n\pi x$$

Integrate this series term by term to obtain the series

$$g(x) = C - 2 \sum_{n=1}^{\infty} (n\pi)^{-2} \cos n\pi x$$

where C denotes a constant of integration. Find C such that $g(x) = x - x^2/2$. *Hint:*

$$\sum_{n=1}^{\infty} n^{-2} = \tfrac{1}{6}\pi^2$$

17. Compute the Fourier cosine series coefficients for the function $g(x) = x - x^2/2$ defined on $0 < x < 1$ directly and compare with the results of Exercise 16.

18. Compute the Fourier cosine series coefficients for the function

$$h(x) = 2/\pi \sin \pi x/2, \qquad 0 < x < 1$$

If this series is differentiated term by term, does the resulting series converge to $h'(x) = \cos \pi x/2$?

2.2 GENERALIZED FOURIER SERIES

We show in this section that the Fourier series expansion of a square-integrable function in terms of sines and cosines is really an abstract version of an operation that is already familiar to many of the readers of this book. We are thinking of the operation of writing an arbitrary vector in R^2 or R^3 in terms of a basis. Engineering texts frequently refer to the basis vectors in R^3 as **i**, **j**, and **k**. Then the vector

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

denotes the directed line segment from $(0, 0, 0)$ to (a, b, c). We begin by recalling a few salient facts about R^N.

Vector Space R^N

Very early in the study of the applications of mathematics, one encounters the techniques of vector algebra. Often vectors are first presented as directed line segments, and the operations of forming linear combinations of vectors are carried out geometrically by means of the so-called parallelogram law. Such methods are inconvenient for computational purposes and do not readily generalize. This motivates the introduction of an orthonormal basis and the description of vectors in terms of components. In this setting, vectors are generally written as N-tuples of numbers $\mathbf{x} = (x_1, \ldots, x_N)$. We may interpret this to mean the following:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_N\mathbf{e}_N$$