Chapter 3

Order of Accuracy of Finite Difference Schemes

3.1 Order of Accuracy

n the previous two chapters we classified schemes as acceptable or nonaccept-
ble only on the basis of whether or not they are convergent. This, via the
ax-Richtmyer equivalence theorem, led us to consider stability and consistency.
However, two convergent schemes may differ considerably in how well their so-
utions approximate the solution of the differential equation. This may be seen
y comparing Figures 1.4 and 1.6, which show solutions computed with the Lax-
riedrichs and leapfrog schemes. Both of these schemes are convergent for λ
qual to 0.8, yet the leapfrog scheme has a solution that is closer to the solution
f the differential equation than does the Lax-Friedrichs scheme. In this section
e define the order of accuracy of a scheme, which can be regarded as an exten-
ion of the definition of consistency. The leapfrog scheme has a higher order of
ccuracy than does the Lax-Friedrichs scheme, and thus, in general, its solutions
ill be more accurate than those of the Lax-Friedrichs scheme. The proof that
chemes with higher order of accuracy generally produce more accurate solutions
s in Chapter 10.

Before defining the order of accuracy of a scheme, we introduce two schemes, hich, as we will show, are more accurate than most of the schemes we have resented so far. We will also have to pay more attention to the way the forcing unction, $f(t,x)$, is incorporated into the scheme.

The Lax-Wendroff Scheme

To derive the Lax-Wendroff scheme (see [31]) we begin by using the Taylor series n time for $u(t+k,x)$, where u is a solution to the inhomogeneous one-way ave equation

$$u(t+k,x) = u(t,x) + ku_t(t,x) + \frac{k^2}{2}u_{tt}(t,x) + O(k^3).$$

We now use the differential equation that u satisfies,

$$u_t = -au_x + f$$

and the relation

$$u_{tt} = -au_{tx} + f_t = a^2u_{xx} - af_x + f_t,$$

to obtain

$$u(t+k,x) = u(t,x) - ak\,u_x(t,x) + \frac{a^2k^2}{2}u_{xx}(t,x) + kf - \frac{ak^2}{2}f_x + \frac{k^2}{2}f_t + O(k^3).$$

Replacing the derivatives in x by second-order accurate differences and f_t by a forward difference, we obtain

$$\begin{aligned} u(t+k,x) =& u(t,x) - ak\frac{u(t,x+h)-u(t,x-h)}{2h} \\ &+ \frac{a^2k^2}{2}\,\frac{u(t,x+h)-2u(t,x)+u(t,x-h)}{h^2} \\ &+ \frac{k}{2}\left[f(t+k,x)+f(t,x)\right] - \frac{ak^2}{2}\,\frac{\left[f(t,x+h)-f(t,x-h)\right]}{2h} \\ &+ O(kh^2) + O(k^3). \end{aligned}$$

This gives the Lax-Wendroff scheme

$$\begin{aligned} \frac{v_m^{n+1}-v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2k}{2}\,\frac{(v_{m+1}^n - 2v_m^n + v_{m-1}^n)}{h^2} \\ = \tfrac{1}{2}(f_m^{n+1}+f_m^n) - \frac{ak}{4h}(f_{m+1}^n - f_{m-1}^n), \end{aligned} \tag{3.1.1}$$

or, equivalently,

$$\begin{aligned} v_m^{n+1} =& v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + \frac{a^2\lambda^2}{2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n) \\ &+ \frac{k}{2}(f_m^{n+1}+f_m^n) - \frac{ak\lambda}{4}(f_{m+1}^n - f_{m-1}^n) \end{aligned} \tag{3.1.2}$$

where $f_m^n = f(t_n, x_m)$.

The Crank-Nicolson Scheme

To derive the Crank-Nicolson scheme we begin with the formula

$$u_t = \frac{u(t+k,x)-u(t,x)}{k} + O(k^2)$$