Homework #3 • MATH 419 • The Lemma of Riemann & Lebesgue

- submit your write-up Wednesday 01 June.

A) Riemann-Lebesgue Lemma (3 pages max, 5 extra points based on quality of presentation.) Present a rigorous proof for a variant of the Riemann-Lebesgue lemma

$$\lim_{k\to\infty} \int_a^b f(x)\, \cos k(x-\alpha)\; dx \;\; = \;\; 0 \qquad (1)$$

where $f(x)$ is a continuous function on $[a, b]$, and α is a given constant. Unlike the case in lecture, $f(x)$ is not necessarily periodic, but the proof requires only a minor modification. As a guide, your proof should include the following ideas:

- Break the interval into three pieces, two (quite) small, the other large. You may bound the contributions of each by $\epsilon/2$.
- The continuity argument will apply to the larger subinterval. Use the rigorous definition of continuity in your integral bound.
- The existence of a lower bound for the index, $k > N(\epsilon)$, is thus detemined from the bounding of the large interval alone.

Discuss how this proof essentially extends the Riemann-Lebesgue lemma to all piecewise continuous (finite number of discontinuities) functions on $[A, B]$. (With a bit of Lebesgue's measure theory, this is further extended to continuous almost everywhere — but you needn't consider this.)

B) Decay (2 pages max) Consider the $f_o(x)$ in the example 1.13 in BN. Apply integration by parts once to identify the slowest-decaying, algebraic behaviour of the full-period Fourier coefficients. Make a comparison plot to see how much the slowest decaying part reflects the behaviour of the exact coefficients (which are provided in the text).