Homework #8 • MATH 419 • Last Homework

- submit your write-up on **Thursday** 28 July (my parents arrive on Friday).

A) A Bounded Linear Operator (4 pages) Consider the operator

$$K[f] = \int_0^x (x-y)\, f(y)\, dy \tag{1}$$

where $f(x) \in L_2[0,1]$. Prove that $K \in B(L_2, L_2)$ with $||K||^2 \le 1/12$. You might first need to verify the identity

$$\left(\int_0^x (x-y)\, f(y)\, dy \right)^2 \;\le\; \left(\int_0^x (x-y)^2\, dy \right) \left(\int_0^x |f(y)|^2\, dy \right) \tag{2}$$

- Use an induction argument to obtain the power operator formula

$$K^n[f] = \frac{1}{(2n-1)!} \int_0^x (x-y)^{(2n-1)}\, f(y)\, dy \tag{3}$$

then apply this result to find a very pretty expression for the solution $f(x)$ to the integral equation

$$f(x) - \int_0^x (x-y)\, f(y)\, dy \;=\; g(x) \tag{4}$$

given $g(x) \in L_2[0,1]$. (Pretty means no summation — one integration is inevitable.) Check your calculations by choosing any particular $g(x)$ and verifying that your general solution formula satisfies the integral equation.

bonus: The norm of $||K||$ can be evaluated by clear thinking & some simple calculus.