MATH 462 - Homework #1 Solution - by Ben Ong

Derivation of the Euler Equations

We present a first principles derivation of the Euler Equations for two-dimensional fluid flow in three-dimensional cylndrical cooordinates (r, θ, z). We assume no vertical flow ($u_z = 0$) and no vertical variations ($\partial/\partial z \equiv 0$). We define $u_r(r, \theta, t)$ and $u_\theta(r, \theta, t)$ to be flow components in the $\hat{e}_r$ and $\hat{e}_\theta$ directions respectively.

Conservation of mass tells us that the change of mass inside a control volume has to equal the net flux of mass through the control volume. The change in mass due to time variation in density and volume is given by

$$\big[\rho(r,\theta,t+\Delta t) - \rho(r,\theta,t)\big]\Delta r(r\Delta\theta)\Delta z \approx \frac{\partial\rho}{\partial t}\Delta t\Delta r(r\Delta\theta)\Delta z \tag{1}$$

Denoting the flux through surfaces with $\hat{e}_r(r, \theta, z)$ and $\hat{e}_r(r + \Delta r, \theta, z)$ normal vectors as

$$\begin{aligned}\text{flux}^r &= \rho(r,\theta,t)\big(u_r(r,\theta,t)\Delta t\big)(r\Delta\theta)\Delta z\\ \text{flux}^{r+\Delta r} &= \rho(r+\Delta r,\theta,t)\big(u_r(r+\Delta r,\theta,t)\Delta t\big)\big((r+\Delta r)\Delta\theta\big)\Delta z\end{aligned}$$

We see that the net flux in the $\hat{e}_r$ is

$$(\text{net flux})^r \approx \frac{\partial(\rho u_r r)}{\partial r}\Delta r\Delta t\Delta\theta\Delta z \tag{2}$$

Similarly

$$\begin{aligned}\text{flux}^\theta &= \rho(r,\theta,t)\big(u_\theta(r,\theta,t)\Delta t\big)\Delta r\Delta z\\ \text{flux}^{\theta+\Delta\theta} &= \rho(r,\theta+\Delta\theta,t)\big(u_\theta(r,\theta+\Delta\theta,t)\Delta t\big)\Delta r\Delta z\end{aligned}$$

Thus

$$(\text{net flux})^\theta \approx \frac{\partial(\rho u_\theta)}{\partial\theta}\Delta\theta\Delta t\Delta r\Delta z \tag{3}$$

and

$$(\text{net flux})^z \equiv 0 \tag{4}$$

Combining the four above equations give the conservation of mass condition

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$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial\theta} = 0 \tag{5}$$

Newton's Law tells us that the net change in momentum is equal to the net impulse. The net change in momentum is affected by (i) momentum flux leaving through the control volume, and through (ii) time variation in density and volume of the control box.

The momentum is given by the product of the momentum density and the volume.

$$
\begin{aligned}
\vec{m}(t) &= \rho(r,\theta,t)\big[u_r(r,\theta,t)\hat{e}_r(\theta) + u_\theta(r,\theta,t)\hat{e}_\theta(\theta)\big]\big(\Delta r(r\Delta\theta)\Delta z\big) \\
\vec{m}(t+\Delta t) &= \rho(r,\theta,t+\Delta t)\big[u_r(r,\theta,t+\Delta t)\hat{e}_r(\theta) + u_\theta(r,\theta,t+\Delta t)\hat{e}_\theta(\theta)\big]\big(\Delta r(r\Delta\theta)\Delta z\big)
\end{aligned}
$$

thus the contribution to the net change from (ii) is

$$
\vec{m}^t \approx \frac{\partial}{\partial t}\Big\{\rho\big[u_r\hat{e}_r + u_\theta\hat{e}_\theta\big]\Big\}\big(\Delta t\Delta r(r\Delta\theta)\Delta z\big) \tag{6}
$$

Denoting the flux through surfaces with $\hat{e}_r(r,\theta,z)$ and $\hat{e}_r(r+\Delta r,\theta,z)$ normal vectors as

$$
\begin{aligned}
\vec{m}(r) &= \rho(r,\theta,t)\big[u_r(r,\theta,t)\hat{e}_r(\theta) + u_\theta(r,\theta,t)\hat{e}_\theta(\theta)\big]\big((u_r(r,\theta,t)\Delta t)(r\Delta\theta)\Delta z\big) \\
\vec{m}(r+\Delta r) &= \rho(r+\Delta r,\theta,t)\big[u_r(r+\Delta r,\theta,t)\hat{e}_r(\theta) + u_\theta(r+\Delta r,\theta,t)\hat{e}_\theta(\theta)\big] \\
&* \Big(\big(u_r(r+\Delta r,\theta,t)\Delta t\big)\big((r+\Delta r)\Delta\theta\big)\Delta z\Big)
\end{aligned}
$$

Thus the net flux out of the two surfaces is approximated by taking

$$
\lim_{r\to 0}\frac{\vec{m}(r+\Delta r) - \vec{m}(r)}{\Delta r} = \vec{m}^r \approx \frac{\partial}{\partial r}\Big\{\rho r u_r\big[u_r\hat{e}_r + u_\theta\hat{e}_\theta\big]\Big\}(\Delta t\Delta r\Delta\theta\Delta z) \tag{7}
$$

Similarly

$$
\begin{aligned}
\vec{m}(\theta) &= \rho(r,\theta,t)\big[u_r(r,\theta,t)\hat{e}_r(\theta) + u_\theta(r,\theta,t)\hat{e}_\theta(\theta)\big]\big((u_\theta(r,\theta,t)\Delta t)\Delta r\Delta z\big) \\
\vec{m}(\theta+\Delta\theta) &= \rho(r,\theta+\Delta\theta,t)\big[u_r(r,\theta+\Delta\theta,t)\hat{e}_r(\theta+\Delta\theta) + u_\theta(r,\theta+\Delta\theta,t)\hat{e}_\theta(\theta+\Delta\theta)\big] \\
&* \Big(\big(u_\theta(r,\theta+\Delta\theta,t)\Delta t\big)\Delta r\Delta z\Big)
\end{aligned}
$$

From vector calculus,

$$
\hat{e}_r(\theta+\Delta\theta) = \hat{e}_r(\theta) + \Delta\theta\hat{e}_\theta(\theta) \tag{8}
$$
$$
\hat{e}_\theta(\theta+\Delta\theta) = \hat{e}_\theta(\theta) - \Delta\theta\hat{e}_r(\theta) \tag{9}
$$

Substituting into $\vec{m}(\theta+\Delta\theta)$ gives

$$
\begin{aligned}
\vec{m}(\theta+\Delta\theta) &= \rho(r,\theta+\Delta\theta,t)\big[u_r(r,\theta+\Delta\theta,t)(\hat{e}_r(\theta)+\Delta\theta\hat{e}_\theta(\theta)) + u_\theta(r,\theta+\Delta\theta,t)(\hat{e}_\theta(\theta)-\Delta\theta\hat{e}_r(\theta))\big] \\
&* \Big(\big(u_\theta(r,\theta+\Delta\theta,t)\Delta t\big)\Delta r\Delta z\Big)
\end{aligned}
$$

Thus

$$\begin{aligned}
\frac{\vec{m}(\theta+\Delta\theta)-\vec{m}(\theta)}{\Delta t\Delta r\Delta z} &= \big[\rho(r,\theta+\Delta\theta,t)u_r(r,\theta+\Delta\theta,t)u_\theta(r,\theta+\Delta\theta,t)-\rho(r,\theta,t)u_r(r,\theta,t)u_\theta((r,\theta,t)\big]\hat{e}_r \\
&+ \big[\rho(r,\theta+\Delta\theta,t)u_\theta(r,\theta+\Delta\theta,t)u_\theta(r,\theta+\Delta\theta,t)-\rho(r,\theta,t)u_\theta(r,\theta,t)u_\theta((r,\theta,t)\big]\hat{e}_\theta \\
&+ \big[\rho(r,\theta+\Delta\theta,t)u_r(r,\theta+\Delta\theta,t)u_\theta(r,\theta+\Delta\theta,t)\big]\Delta\theta\hat{e}_\theta \\
&- \big[\rho(r,\theta+\Delta\theta,t)u_\theta(r,\theta+\Delta\theta,t)u_\theta(r,\theta+\Delta\theta,t)\big]\Delta\theta\hat{e}_r
\end{aligned}$$

Dividing both sides by $\Delta\theta$ and taking $\lim \Delta\theta \to 0\frac{\vec{m}(\theta+\Delta\theta)-\vec{m}(\theta)}{\Delta\theta}$ gives

$$\vec{m}^\theta \approx \left[\frac{\partial}{\partial\theta}\big\{\rho u_\theta(u_r\hat{e}_r+u_\theta\hat{e}_\theta)\big\}+\rho u_\theta(u_r\hat{e}_\theta-u_\theta\hat{e}_r)\right]\Delta t\Delta r\Delta z\Delta\theta \tag{10}$$

Combining equations (6), (7) and (10) gives the net change in momentum

$$\begin{aligned}
\left[r\frac{\partial}{\partial t}\{\rho u_r\}\right. &+ \left.\frac{\partial}{\partial r}\{\rho r u_r u_r\}+\frac{\partial}{\partial\theta}\{\rho u_r u_\theta\}-\rho u_\theta^2\right](\Delta r\Delta\theta\Delta z\Delta t)\hat{e}_r \\
&+ \left[r\frac{\partial}{\partial t}\{\rho u_\theta\}+\frac{\partial}{\partial r}\{\rho r u_\theta u_r\}+\frac{\partial}{\partial\theta}\{\rho u_\theta u_\theta\}+\rho u_r u_\theta\right](\Delta r\Delta\theta\Delta z\Delta t)\hat{e}_\theta
\end{aligned} \tag{11}$$

If you expand the derivatives, and impose conservation of mass (equation (5)), the net change in momentum simplifies to (djm: the two *extra* terms come from the CV geometry (8) & (9))

$$\begin{aligned}
\left[\rho\frac{\partial u_r}{\partial t}\right. &+ \left.\rho u_r\frac{\partial u_r}{\partial r}+\frac{1}{r}\rho u_\theta\frac{\partial u_r}{\partial\theta}-\frac{\rho u_\theta^2}{r}\right](r\Delta r\Delta\theta\Delta z\Delta t)\hat{e}_r \\
&+ \left[\rho\frac{\partial u_\theta}{\partial t}+\rho u_r\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\rho u_\theta\frac{\partial u_\theta}{\partial\theta}+\frac{\rho u_r u_\theta}{r}\right](r\Delta r\Delta\theta\Delta z\Delta t)\hat{e}_\theta
\end{aligned} \tag{12}$$

We now need to calculate the net impulse on the system in order to derive the remaining Euler Equations. There are two contributions; (i) From a given body force density $\vec{F}(r,\theta,t) = F^r(r,\theta,t)\hat{e}_r + F^\theta(r,\theta,t)\hat{e}_\theta + F^z(r,\theta,t)\hat{e}_z$ and (ii) from the internal pressure.

The impulse contribution from (i) is simply the force density*volume*Δt.

$$\vec{F}(r,\theta,t) = (F^r(r,\theta,t)\hat{e}_r + F^\theta(r,\theta,t)\hat{e}_\theta + F^z(r,\theta,t)\hat{e}_z)(r\Delta r\Delta\theta\Delta z)\Delta t \tag{13}$$

The force = (pressure*area) exerted on the surfaces with $\hat{e}_r(r,\theta,z)$ and $\hat{e}_r(r+\Delta r,\theta,z)$ normal vectors is

$$\begin{aligned} F_P^r &= P(r,\theta,t)\big[(r\Delta\theta)\Delta z\big]\hat{e}_r(\theta) \\ F_P^{r+\Delta r} &= P(r+\Delta r,\theta,t)\big[(r+\Delta r)\Delta\theta\Delta z\big]\hat{e}_r(\theta) \end{aligned}$$

Thus, the net force contribution from the two surfaces

$$\begin{aligned} \{F_P\}^r &\approx \frac{\partial}{\partial r}\{Pr\}[\Delta r\Delta\theta\Delta z]\hat{e}_r(\theta) \\ &\approx \left[r\frac{\partial P}{\partial r}+P\right][\Delta r\Delta\theta\Delta z]\hat{e}_r(\theta) \end{aligned} \tag{14}$$

Similarly

$$\begin{aligned} F_P^\theta &= P(r,\theta,t)\big[\Delta r\Delta z\big]\hat{e}_\theta(\theta) \\ F_P^{\theta+\Delta\theta} &= P(r,\theta+\Delta\theta,t)\big[\Delta r\Delta z\big]\hat{e}_\theta(\theta+\Delta\theta) \end{aligned}$$

Using equation (9) gives

$$F_P^{\theta+\Delta\theta} = P(r,\theta+\Delta\theta,t)\big[\Delta r\Delta z\big]\big(\hat{e}_\theta(\theta)-\Delta\theta\hat{e}_r(\theta)\big)$$

Thus, the contribution from the two surfaces gives

$$\{F_P\}^\theta \approx \left[\frac{\partial P}{\partial\theta}\ \hat{e}_\theta(\theta)-P\hat{e}_r(\theta)\right]\Delta r\Delta z\Delta\theta \tag{15}$$

And the contribution from surfaces with $\hat{e}_z$ as normals is

$$\{F_P\}^z \equiv 0 \tag{16}$$

Thus the net impulse is (djm: note the amazing cancellation of the non-gradient P term)

$$\big[F^r\hat{e}_r+F^\theta\hat{e}_\theta+F^z\hat{e}_z+\frac{\partial P}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial P}{\partial\theta}\hat{e}_\theta\big]r\Delta r\Delta\theta\Delta z\Delta t \tag{17}$$

Combining equations (12) and (17) gives us Newton's Law

$$(\hat{e}_r) \qquad \frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}=\frac{1}{\rho}\left[-\frac{\partial P}{\partial r}+F^r\right] \tag{18}$$

$$(\hat{e}_\theta) \qquad \frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}=\frac{1}{\rho}\left[-\frac{1}{r}\frac{\partial P}{\partial\theta}+F^\theta\right] \tag{19}$$

$$(\hat{e}_z) \qquad 0=F^z \qquad \text{(imposes condition on body force)} \tag{20}$$

Rotational Flows

Given $\vec{u} = (-\Omega y, \Omega x, 0)$, find the pressure which produces a flow solution to the incompressible Euler Equations with $\vec{F} = -\rho g \hat{z}$

Incompressible Euler Equations

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 & (21) \\ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} &= -\frac{\vec{\nabla} P}{\rho_o} + \frac{\vec{F}}{\rho_o} & (22) \end{aligned}$$

We notice that equation (21) is automatically satisfied. Expanding (22)

$$\begin{aligned} u_t + (uu_x + vu_y + wu_z) &= -\frac{P_x}{\rho_o} \\ v_t + (uv_x + vv_y + wv_z) &= -\frac{P_y}{\rho_o} \\ w_t + (uw_x + vw_y + ww_z) &= -\frac{P_z}{\rho_o} - g \end{aligned}$$

Substituting $\vec{u} = (-\Omega y, \Omega x, 0)$, we get

$$\begin{aligned} -\Omega^2 x &= -\frac{P_x}{\rho_o} & (23) \\ -\Omega^2 y &= -\frac{P_y}{\rho_o} & (24) \\ P_z &= -\rho_o g & (25) \end{aligned}$$

Solving (23) gives

$$P = \frac{\rho_o \Omega^2 x^2}{2} + f(y, z) \tag{26}$$

Differentiating (26) and comparing with (24) gives

$$P_y = f'(y, z) = \rho_o \Omega^2 y \tag{27}$$

Solving (27) gives

$$P = \frac{\rho_o \Omega^2 (x^2 + y^2)}{2} + g(z) \tag{28}$$

Differentiating (28) and comparing with (25) and solving gives

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$$P = \frac{\rho_o \Omega^2 (x^2 + y^2)}{2} - \rho_o g z + constant \tag{29}$$

[djm:] Since the fluid has vorticity, $\nabla \times \vec{u} = (0, 0, 2\Omega) \neq \vec{0}$, the Bernoulli theorem for irrotational flow does not apply. However, the flow is steady and so the Bernouilli theorem for streamlines does apply, but the Bernouilli function can have different constant values on different streamlines (circles around the axis of rotation) and cannot be used to infer the surface geometry.

An astronomer could make a liquid mirror telescope by spinning mercury on a a parabolic surface. Actually, this is being done at a UBC research station in maple Ridge; they have a 6m diameter mirror! They need the correct angular velocity to get a uniform coating of the mercury