Homework #2 • MATH 462 • Simple Incompressible Flows

- please respect page limits.
- submit your write-up Wednesday 30 January.
- remember that the class e-mail is open for discussion.
- please acknowledge collaborations & assistance from colleagues.

A) Expanding Bubble (3 pages, 10pts) It was asked in lecture whether there was a useful flow theory in one space dimension – the answer was no, if in Cartesian coordinates. However, this is not the case in spherical coordinates.

Consider a fluid having infinite spatial extent in three dimensions; assume this fluid to be incompressible and that gravity is absent. A purely radial flow is being generated by a pulsating sphere, centred at the origin, with a time-periodic radius $R(t)$

$$R(t) = 1 + a\sin(\sigma t) \qquad \text{with } 0 < a < 1$$

Extract the Euler equations in spherical coordinates from Acheson (Appendix A7, setting $\nu = 0$) and reduce to the equations relevant for the flow variables $U(r,t)$ and $p(r,t)$. Solve for the flow using the boundary conditions

$$p(r \to \infty) = p^{\infty} \quad ; \quad U(r = R(t)) = dR/dt = R'(t)$$

and explain why the second boundary condition is appropriate.

bonus: (1 page, 5pts) Modify one of the plotting codes to plot the pressure at the sphere's surface as a function of time (and perhaps, varying parameters a, σ). Compare the phase of the expansion with the phases of the two contributions to the pressure (linear and nonlinear) and give an intuitive hypothesis for these results. (Try *help subplot* in matlab.)

B) Steady Streamlines (3 pages, 10pts) Consider an *axisymmetric* flow in three-dimensions defined via the *Stokes streamfunction* $\psi(r,z)$

$$\psi(r,z) = \frac{A}{2}r^2 + \frac{m}{4\pi}\left(1 - \frac{z}{\sqrt{r^2+z^2}}\right)$$

where the flow velocities are given by the derivative relations

$$U = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad ; \quad W = +\frac{1}{r}\frac{\partial \psi}{\partial r} \ .$$

Verify that the above flow satisfies the continuity equation. Then show that the streamfunction ψ is constant along streamlines by verifying that

$$\frac{D\psi}{Dt} = 0$$

and explain why this is not a difficult calculation. It is then straightforward to use the *contour* command in matlab to produce a graphic illustrating the flow (give parameters).

bonus: What is the corresponding pressure field?