Homework #3 • MATH 462 • Vorticity

- please respect page limits.
- submit your write-up Wednesday 06 February.
- remember that the class e-mail is open for discussion.
- please acknowledge collaborations & assistance from colleagues.

A) A Patch of Vorticity (4 pages, 20pts) Consider an incompressible 2D fluid whose initial condition is characterized by a circular patch of vorticity

$$\omega(\vec{x},0) = \begin{cases} \dfrac{1}{\pi a^2} & 0 \le r < a \\ 0 & a < r < \infty \end{cases}$$

where ω is the $\hat{z}$-component of vorticity $\vec{\omega}$ and $r = |\,\vec{x}\,|$. Solve for the initial streamfunction $\psi(\vec{x},0)$, and hence, determine the initial flow velocity. Your task is simplified in this geometry since the Poisson PDE for the streamfunction $\psi(\vec{x},0)$ is really just an ODE. Invoke the BCs that the flow is bounded at the origin, and decays to zero as $r \to \infty$. For uniqueness of solution, it is also necessary to impose continuity on the streamfunction and its associated flow at $r = a$. Describe the resulting flow pattern.

Using an argument involving the vorticity equation, describe the time evolution of this flow for $t > 0$. This result, in turn, makes it easy to determine the pressure distribution. Invoke the conditions that the pressure approaches a constant value p^∞ as $r \to \infty$ and is continuous at $r = a$. Explain why it makes sense that the pressure gradient is not zero!

Make a subplot or two to illustrate the salient features of this flow.

Finally, show that there is a limiting streamfunction as the patch parameter $a \to 0$

$$\Psi(\vec{x},0) = \lim_{a \to 0} \; \psi(\vec{x},0) \; .$$

This is commonly known as the *point vortex* of 2D flow.

B) Vorticity & Compressibility (2 pages, 10pts) Present a discussion based upon problem 1.5 in Acheson. Please show more than just algebra.