Homework #2 • MATH 462 • Simple Incompressible Flows

- please respect page limits.
- submit your write-up noon, friday 30 January.
- remember that the class e-mail is open for discussion.
- please acknowledge collaborations & assistance from colleagues.

A) Streamlines & Streamfunctions (4 pages + plot, 10pts) A keen-eyed student making the flow plots for problem **B)** of Homework #0 will have noticed that the contours of constant ψ seemed oddly parallel to the flow arrows everywhere! This is no accident, of course, and this behavior is investigated in the following problem for a different flow example.

In cylindrical coordinates, an *axisymmetric* flow has no dependence on the θ variable. A *Stokes streamfunction*, $\Psi(r,z)$, provides a definition of a steady, axisymmetric flow velocity having the form $\vec{u} = U(r,z)\,\hat{r} + W(r,z)\,\hat{z}$ through the derivative relations:

$$U = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \qquad ; \qquad W = +\frac{1}{r}\frac{\partial \Psi}{\partial r} \ .$$

For the specific case of the Stokes streamfunction

$$\Psi(r,z) = \frac{A}{2}r^2 + \frac{m}{4\pi}\left(1 - \frac{z}{\sqrt{r^2+z^2}}\right)$$

verify that the flow satisfies the incompressibility condition. Next, show that the streamfunction Ψ is constant along streamlines by verifying that

$$\frac{D\Psi}{Dt} = 0$$

and, if possible, explain why this is not a difficult calculation. It is then straightforward to use the *contour* command in matlab to produce a graphic illustrating the flow (annotations should include, at the very minimum, parameters and flow direction arrows).

Use the Bernoulli streamline theorem to calculate the pressure deviation (assume that the pressure far-upstream is uniformly constant). Indicate regions of relatively high and low pressure regions on your flow graphic and verify that they are consistent with the observed flow curvatures.

extra: Is this example illustrative/suggestive of any realistic flow?

B) Pulsing Bubble (3 pages, 10pts) Consider a fluid having infinite spatial extent in three dimensions; assume this fluid to be incompressible and that gravity is absent. A purely radial flow $\vec{u} = U(r,t)\,\hat{r}$ is being generated by a pulsating sphere, centred at the origin, with a time-periodic radius $R(t)$

$$R(t) = 1 + a\sin(\sigma t) \qquad \text{with } 0 < a < 1$$

Extract the Euler equations in spherical coordinates from Acheson (Appendix A7, setting $\nu = 0$) and reduce to the equations relevant for the flow variables $U(r,t)$ and $p(r,t)$. Solve for the flow using the boundary conditions

$$p(r \to \infty) = p^{\infty} \qquad ; \qquad U(r = R(t)) = dR/dt = R'(t)$$

and explain why the second boundary condition is appropriate.

bonus: (1 page, 5pts) Modify one of the plotting codes to plot the pressure at the sphere's surface as a function of time (and perhaps, varying parameters a, σ). Compare the phase of the expansion with the phases of the two contributions to the pressure (linear and nonlinear) and give an intuitive hypothesis for these results. (Try *help subplot* in matlab.)