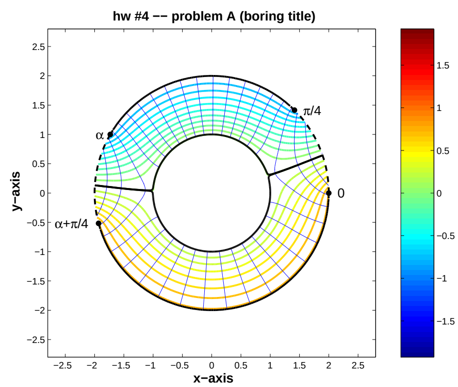
## Homework #4 • MATH 462 • Potential Flow

- submit your write-up noon, Friday 12 February.
- acknowledge collaborations & assistance.
- **this is the major assignment leading up to the midterm.**

**A) Flow Around an Island** (4 pages, 15pts) Apply the method of separation of variables to the Laplace equation to produce all possible velocity potentials, $\phi(r, \theta)$, in the annular region $1 \leq r \leq R$. The BCs are applied as flow velocities normal to the circular boundaries at $r = 1$ and $r = R$. No flow through the inner circle $\hat{r} \cdot \vec{u}(1, \theta) = 0$; and on the outer circle

$$\hat{r} \cdot \vec{u}(R, \theta) = \begin{cases} -1 & \text{for } 0 < \theta < \pi/4 \\ +1 & \text{for } \alpha < \theta < \alpha + \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

which specifies inflow ($0 < \theta < \pi/4$) and outflow ($\alpha < \theta < \alpha + \pi/4$) velocity sectors. Below is a sample plot of the streamlines and contours of velocity potential.




You can download the plotting script *hw04polar.m* with data *hw04data.mat* from the class webpage. The online data only gives a low-quality plot, you are expected to increase the quality for your submission plot(s).

**i)** Present a derivation of the separation of variables solution for $\phi(r, \theta)$ that is organized around the key conceptual steps. Give explicit formulas for all of the coefficients based on the above BCs. How is it that the flow velocity cannot be uniquely determined?

**ii)** Find the corresponding streamfunction (the harmonic conjugate).

**iii)** For your own choice of $\alpha$, make a plot of the streamlines and contours of velocity potential ($R = 2$ makes a nice plot, but seems not to matter) for the unique solution with zero circulation. On your plot, indicate regions of fast/slow flow (include explanations). Also indicate regions of high/low pressure as deduced by flow curvature (again, include logic). Note that my script has some mysterious, but very useful features.

**iv)** For your specific choice of $\alpha$, what fraction of the flow is diverted above, versus below, the island? Explain.

**Extra plots:** Some intriguing behaviour occurs if enough circulation is allowed in the solution. Investigate & explain.

**Bonus:** Your solution method is doomed if the angle of the outlet doesn't match the angle of the inlet. Explain in terms of the fluid physics and the mathematics.

**B) Flow Plots** (1 page + plots, 5pts) Plotting flow solutions with singularities can be tricky in Matlab. Consider two ways of calculating the same streamfunction:

$$
\begin{aligned}
\Phi_1(z) &= \frac{z}{c} + \frac{1}{2\pi}\log\left(\frac{z+c}{z-c}\right) \\
\Phi_2(z) &= \frac{z}{c} + \frac{1}{2\pi}\log(z+c) - \frac{1}{2\pi}\log(z-c)
\end{aligned}
$$

where $c = e^{i\alpha}$ where $\alpha$ is an angle. Describe the flow, and compare the plotted results when $\alpha$ is non-zero. Pay special attention to the $\psi = 0$ streamline. Explain why the Matlab plots of $\Phi_1(z)$ and $\Phi_2(z)$ are different.

**C) Use the Force** (3 pages, 10pts) Read Chapter 4 and present a discussion based upon problem 4.3 in Acheson.