Homework #8 • MATH 462 • Sound Waves & Nonlinear Waves

- submit your write-up Friday 26 March.

A) Music of the Sphere? (3 pages, 10pts) This problem is based on #3.13 in Acheson, and requires reading of section 3.6 of the chapter on waves.

i) Starting from the compressible Euler equations in coordinate-free form (as in equations 3.55 & 3.56) with adiabatic thermodynamics, show that the linearization about a quiet state $\rho = \rho_0$ and $\vec{u} = \vec{0}$ gives a 3D wave equation for the density. Specialize this result for a spherically symmetric wave.

ii) For a spherical shell of radius L, derive the eigenvalue relation for the natural (temporal) frequencies, ω, of the interior standing waves

$$\tan \frac{\omega L}{c} = \frac{\omega L}{c} \ .$$

(The natural frequencies are associated with separation of variables solutions having the form $\rho(r,t) = f(r)\, cos(\omega t)$.) Explain why the boundary conditions of bounded density at the origin and zero velocity at $r = L$ are reasonable choices.

iii) Numerically calculate the first three eigenfrequencies (as multiples of c/L), and comment upon whether or not you think the *music of this sphere* would be a truly harmonious sound.

B) It's a Bore! (3 pages + plot, 15pts) There is a further simplification to our surface wave theory of weeks 7 & 8 that applies when the wavelength of the waves is much greater than the mean depth of the fluid. In 1D, the PDEs that describe this fluid are

$$h_t + (hu)_x = 0 \qquad ; \qquad u_t + u\, u_x + g\, h_x = 0$$

for $u(x,t)$, and the fluid layer depth, $h(x,t)$. The constant g is the acceleration of gravity. These equations are presented as equations 3.96 & 3.97 in Acheson.

i) Following the presentation in lecture, derive the Riemann invariant relations for these PDEs.

ii) Consider a modification of flow suggested by figure 3.17 in Acheson. The initial river has constant height h_0 but seaward flow velocity $-U_0 < 0$. The incoming tidal seawater will then be idealized as a moving wall with velocity $U_s = -U_0 + \alpha t$. Again, following the presentation in lecture, derive the equation for the characteristics that are generated by the moving wall boundary. As the upstream conditions are constant, these characteristics can be shown to be straight lines.

Make a plot of these characteristics in the x-t plane. I used the following values for the constants: $c_0 = \sqrt{gh_0} = 1$, $\alpha = c_0/3$, $U_0 = c_0/4$.