Homework #08 • MATH495/STAT490 • Markov Processes

- submit your write-up before 12 noon on **Thursday** 13 November.
- note page limits, highlight major results.
- please indicate partners in collaborative efforts. Thank you.
- to aid the grader, please begin each lettered problem on a new page.
- **the page limit on part A) has not been set as a guideline for how many pages you should write, but rather to leave you enough space so that matrices and vectors can be written to be comfortably read.**

A) Ehrenfest Model (7 pages, 20 pts) The ultimate goal is to answer the question, *Beginning from the state with all 4 items in one urn, what is the expected number of steps T until all the items are in the other urn?* Denote the states by $s_j = (j, 4-j)$ where $j = 0, 1, 2, 3, 4$.

- use the modified Markov transition matrix for which the $(4,0)$-state is now absorbing (that is, $p_{4k} = 0$ except for $p_{44} = 1$);
- draw the state diagram for this process;
- determine the transition matrix $\mathbf{P}$;
- find the exact eigenvalues of $\mathbf{P}^{\mathrm{T}}$ (you may use the maple worksheet) and show that the result agrees with matlab's numerical output;
- the eigenvectors of $\mathbf{P}^{\mathrm{T}}$ for each eigenvalue λ can be obtained from the formula

$$\vec{v}_\lambda = \begin{pmatrix} (\lambda^2 - 3/4)\lambda(\lambda - 1) \\ (\lambda^2 - 3/8)(\lambda - 1) \\ (3/4)\lambda(\lambda - 1) \\ (3/8)(\lambda - 1) \\ (3/32) \end{pmatrix}$$

  show that the results are consistent with matlab's numerical output;
- use matlab to determine the values a_λ such that

$$\vec{y}^{\,n} = \sum_\lambda a_\lambda \, \lambda^n \, \vec{v}_\lambda$$

- finally note that the final component of $\vec{y}^{\,n}$ is the probability that $T \leq n$ and recall the formula from the 01 October lecture

$$E[T] = \sum_0^\infty P\{T > n\}$$

  this should result in several geometric series which then answers the expected value question.
- **bonus:** where does that eigenvector formula come from?

B) Still More Rain (3 pages, 10 pts) Problem # 46 from Ross, Chapter 4.