Homework #10 • MATH495/STAT490 • Last Homework

- submit your write-up before 12 noon on **Thursday** 27 November.

A) Random Walk I (4 pages, 10 pts) For the one-dimensional random walk, denote the walker's position at step n by s_n. The walker begins in state $s_0 = 0$ and moves with transition probabilities p to the right and q to the left. Define two probabilities at each step: return to origin $u_n = \mathrm{P}\{s_n = 0\}$ and first return to origin $v_n = \mathrm{P}\{s_n = 0 \mid s_k \neq 0,\ k = 1 \ldots n-1\}$. Find a combinatorial expression for u_n and construct its generating function. Argue that the summation is in fact

$$U(z) = \sum_{n=0}^{\infty} u_n\, z^n = (1 - 4pqz^2)^{-1/2}\ . \tag{1}$$

Next, explain why

$$u_n = \sum_{k=1}^{n} v_k\, u_{n-k}\ ,$$

substitute into the summation (1), and then use this result to find the generating function for v_n. Finally, use this second generating function $V(z)$ to comment on the total probability and mean number of steps for the first return to origin.

B) Random Walk II (4 pages max, 10 pts) For the symmetric ($p = q = 1/2$) one-dimensional random walk, denote the number of paths connecting $s_0 = a$ to $s_n = b$ as $N(a, b, n)$ for $a, b > 0$. Now consider the case of *reflecting* random walks, whose paths are restricted to states $s_n \geq 0$. These walks have symmetric left/right transitions, except in the event that $s_n = 0$, where the next state is $s_{n+1} = 1$ with probability 1. Denote the total number of reflecting random walks connecting $s_0 = a$ to $s_n = b$ as $N^r(a, b, n)$. The key property connecting these random walks is the *reflection principle* which says that $N^r(a, b, n) = N(-a, b, n)$. Apply this principle to the following problem.

Suppose that survivor A gets a votes and survivor B gets b, with $a > b$. If the votes are randomly mixed before being revealed, what is the chance that A is always ahead of B during the final exciting vote tally?