Investigation #1 • APMA 900 • Integral Asymptotics

- please respect page limits
- submit your write-up Wednesday 03 October (unless indicated otherwise)
- remember that the class e-mail is open for discussion.
- guidelines for reports are available on the class webpage.

A) (2-3 pages, due 26 September) Present a derivation of the full asymptotic expansion for $\mathrm{erfc}(z)$ as $z \to +\infty$

$$\mathrm{erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi}z}\left\{1 + \sum_1^\infty \frac{a_j}{(2z^2)^j}\right\} \quad \text{as } z \to +\infty.$$

Is the series convergent? Prove the asymptotic nature of this expansion by bounding the magnitude of the residual error. Show that this error after any finite truncation is asymptotically smaller than the final term of the series uniformly over all complex z in the sector $|\arg z| \leq \pi/2 - \delta$, where $0 < \delta \ll 1$.

group challenge (optional): Extend the validity range of the asymptotic expansion to the broader sector $|\arg z| < 3\pi/4$.

B) (2-3 pages) Set up a careful derivation of the full asymptotic expansion for $x \to +\infty$ of the Bessel function $I_0(x)$. Begin from the integral representation

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos t}\, dt\ .$$

You needn't calculate all coefficients explicitly, but your set-up should make it clear how to proceed to all orders. Include the analysis of subdominant contributions.

C) (2 pages) Present a numerical investigation comparing the asymptotic properties as $x \to +\infty$ of: the Matlab intrinsic function *besseli*, finite asymptotic expansions, and Matlab numerical quadrature. Use the Matlab intrinsic function as a baseline. Show & comment upon quantitative error results.

D) (2 pages) Present a formal calculation of the full asymptotic expansion for the integral

$$I(x) = \int_0^1 \sqrt{t}\, e^{ixt}\, dt\ .$$

Follow the suggestions given in the attached copy from Bender & Orszag (pages 277 & 311).

E) (1 page) Show that the leading order stationary phase formula is consistent with the far-field asymptotics of the Airy function.