Investigation #2 • APMA 900 • Integral Asymptotics II

- please respect page limits
- submit your write-up Wednesday 17 October.
- remember that the class e-mail is open for discussion.
- guidelines for reports are available on the class webpage.

A) Max Norm (2 pages): The L_p norm of a real function $g(x)$ on $a \le x \le b$ is defined by the integral

$$||g||_p = \left(\int_a^b |g(x)|^p \, dx \right)^{\frac{1}{p}} .$$

Assuming $g(x)$ is sufficiently continuous, use a formal Laplace argument to explain the terminology *max norm* associated with the asymptotic limit

$$||g||_\infty \sim ||g||_p \qquad \text{as } p \to \infty .$$

Note that the relevant Taylor expansion is that of $\ln |g(x)|$. Address the situations of: (i) single, isolated interior maxima; (ii) non-stationary endpoint maxima; (iii) multiple, but unequal, interior maxima; and (iv) multiple, but equal, interior maxima.

B) Hankel Function (5 pages) Present a steepest descent derivation for the first few terms of the asymptotic expansion for $r \to +\infty$ of the Hankel function $H_0^1(r)$. Begin from the integral representation

$$H_0^1(r) = -\frac{2i}{\pi} \int_{\mathcal{C}} e^{ir\cosh k} \, dk$$

where the contour $\mathcal{C}$ goes from $k = 0$ to $Re(k) \to +\infty$ with $Im(k) = \pi/2$. This is half of the *Sommerfeld* contour; the full contour begins from $Re(k) \to -\infty$ with $Im(k) = -\pi/2$ and introduces a factor of 1/2.

Also show that the limit as $R \to +\infty$ of the integral from $k = R$ to $k = R + i\pi/2$ vanishes by a modified Jordan argument. (I found the substitution $m = e^k$ to be useful.)

C) Numerical Steep<u>ish</u> Descent (2 pages) Design a quantitative comparison of numerical quadratures for the above Hankel integral following contours: (i) along the positive real axis; (ii) a (rough) piecewise linear approximation of $\mathcal{C}$; and (iii) a (reasonable) continuous parametric approximation of $\mathcal{C}$. Note that there are two numerical controls at work here, the quadrature tolerance and the use of a finite upper limit.

optional: Design a version of (iii) that maps onto a finite range of integration.

D) Series Solution (2 pages) Construct a solution $y(x)$ to the ODE

$$y'' + 2xy' = 0 \qquad \text{with } y(+\infty) = y'(+\infty) = 0$$

by hypothesizing the solution form $y(x) = f(x)\, e^{-x^2}$. Derive the ODE for the function $f(x)$ and develop a series solution in inverse powers of x. Comment upon the resulting expression.