Investigation #3 • APMA 900 • Perturbation Theory

- submit your scariest write-up Wednesday 31 October.
- guidelines for reports are available on the class webpage.

A) Three Solutions (3-4 pages): Consider the nonlinear ODE initial value problem for $y(t)$:

$$y' = \cosh \epsilon y \sim 1 + \epsilon^2 \frac{y^2}{2} + \epsilon^4 \frac{y^4}{24} + \ldots \qquad ; \qquad y(0) = 0$$

where ϵ is a small, real parameter. Carry out the following three solution approaches, and present substantiated conclusions from your results as directed by the questions posed. The quality of your discussion is primary for this problem.

This first-order ODE possesses a closed form exact solution. What is the domain of existence for this IVP solution? (Integral tables for this and the perturbation solutions will be useful.)

Construct a solution represented by the straightforward three-term perturbation expansion

$$y(t) \sim y_0(t) + \epsilon^2 y_1(t) + \epsilon^4 y_2(t) + \ldots \qquad \text{as } \epsilon \to 0 \; .$$

What is the asymptotic error of this three-term expansion at $O(1)$ times? What is the domain of existence of this truncated representation? What is the domain of asymptotic validity (give thorough reasons)?

Construct a second, rather unusual, two-term perturbation expansion

$$y(t) \sim Y_0(t;\epsilon) + Y_1(t;\epsilon) + \ldots \qquad ; \qquad Y_0 \gg Y_1 \qquad \text{as } \epsilon \to 0$$

where the leading order $Y_0(t;\epsilon)$ is determined by keeping the first *two* Taylor terms of the hyperbolic cosine expansion. The first correction can be reduced to quadrature by noticing that $1/Y_0'$ acts as an integrating factor for the Y_1 defining equation. (Hint: The derivative of the Y_0 equation is a very useful expression.) What is the asymptotic error of this two-term expansion at $O(1)$ times? What is the domain of existence of this truncated representation? What is the domain of asymptotic validity (again, give careful reasons)?

Bonus: I was sufficiently surprised at the success of the last approach that I had to do a computational check.

B) Orbits (4 pages) Present a Poincaré-Linstedt analysis (as developed in lecture) for the ODE as posed in Exercise #1 in Holmes, page 128. Carry out the analysis through second correction; this should allow you to address the formula suggested by b) part. Compare polar plots of $(r(\theta), \theta)$ for various orders of the approximation (choose α appropriately). For historical significance, do the calculation as described in c) part. (I don't really care about d) part.)

C) Eigenvalue Degeneracies (3 pages) A nice interpretation of the Jordan canonical form can be found in Appendix B of Strang's book on linear algebra. So, if a matrix $[A_0]$ has a double eigenvalue λ with algebraic multiplicity 2, but only geometric multiplicity 1, then there are vectors $\vec{w_1}$ and $\vec{w_2}$ which satisfy

$$[A_0]\vec{w_1} = \lambda\, \vec{w_1} \qquad ; \qquad [A_0]\vec{w_2} = \lambda\, \vec{w_2} + \vec{w_1} \; .$$

Use this fact to derive a three-term asymptotic expansion for the breaking of the degeneracy for the perturbed matrix $[A_0 + \epsilon\, A_1]$.

Confirm the results of the theory by illustrating through a specific example. There is no need to repeat the derivation, just present the relevant quantities in the general framework you have developed. (Hint: working the example first will give you an idea about what goes wrong & what needs to be fixed.)