Investigation #4 • APMA 900 • Perturbation Theory II

- submit your write-up by Wednesday 14 November.

A) Waves in Two Space Dimensions (4 pages): For a linear wave equation

$$u_{tt} - c^2(y)\,\nabla^2 u = u_{tt} - c^2(y)\,(u_{xx} + u_{yy}) = 0$$

with an inhomogeneous wavespeed $c^2(y)$, consider wave solutions $u(x, y, t)$ having the form

$$u(x, y, t) = v(y)\; e^{i\omega(x\,\sin\phi - t)}$$

where ϕ is a real parameter. Interpret the angle ϕ, in the case that the wavespeed is a constant ($c = 1$), by constructing the wave solution which propagates with *phase speed* in the positive y-direction.

Now, consider the inhomogeneous case where the wavespeed is of the form

$$c^2(y) = \frac{1}{n^2(y)} = \frac{8}{7 - \tanh(y)}$$

and the temporal frequency $\omega \gg 1$. A solution for $v(y)$ can be constructed following the method developed in the lecture for the *slowly-varying oscillator*, except that here the large ω has the equivalent role as the small ϵ. Choose the wave solution which resembles the above $c = 1$ wave as $y \to -\infty$, and show how to carry out the asymptotic expansion to first correction in the phase. (Ignore the phenomenon of backscattering.)

Produce plots of real$(u(x, y, t))$ at various times, to visualize a basic physical phenomenon that is usually introduced in high school science classes. Explain. What additional understanding of this phenomenon is provided by this asymptotic analysis? What restrictions are there (if any) on this solution construction?

Bonus: Provide numerical convergence results for the asymptotic ODE solve.

B) Limit Cycles (3 pages) Present a perturbation analysis for small ϵ solutions $y(t)$ of the ODE

$$y'' - \epsilon\,\sin y' + y = 0 \qquad ; \qquad y(0) = 0 \quad \& \quad y'(0) = O(1)$$

based upon the amplitude/phase construction developed in lecture. Begin from a solution representation having the form

$$y(t) = Y[a(t), \tau(t)] = a\sin\tau + Y_1[a, \tau]$$

which is 2π-periodic in τ and $Y_1 \ll O(1)$. The variable $\tau(t)$ is a reparametrization of time. Present a systematic derivation of asymptotic slow-time ODEs

$$\begin{aligned} a_t &\sim A_1[a, \tau_t] &&= o(1) \\ \tau_t &\sim 1 + T_1[a, \tau_t] &&= 1 + o(1) \end{aligned}$$

as $\epsilon \to 0$. (Note the use of square brackets technically denotes a *functional* dependence, which could involve derivatives of a, τ or τ_t). Also, certain Bessel identities (equations 9.1.41-5 in my Abramowitz & Stegun) are necessary for solving this problem. Use your asymptotic results to explain Matlab computations of the solutions (*code22.m* on the webpage).

C) Sturm-Liouville (4 pages) Show the solution process for the self-adjoint Sturm-Liouville ODE problem

$$\mathcal{L}[y(t)] = (t^2\, y')' = 1 \qquad ; \qquad y(1) = 0 \quad \& \quad y(2) = 0$$

which results in the exact expression

$$y(t) = \ln\left(t\; 4^{(1/t\,-\,1)}\right) \; .$$

Use the fact that the operator $\mathcal{L}$ is of Euler type to find a complete set of eigenfunctions. That is, try solutions of the form t^p; also note that the case where $p = -1/2$ must be treated differently.

The general solution of the Sturm-Liouville problem can be written as an expansion over all eigenfunctions. Verify using numerical quadrature that finite expansions using the above eigenfunctions indeed converge to the known exact solution.

Bonus: Show how the convergence of an expansion for the forcing is markedly different from the solution (Gibb's phenomenon).