Class Notes for APMA935 Mathematical Modelling

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Sturm-Liouville Eigenvalue Problem

A. Introduction

Consider the following functionals

$$I[u] = \int_a^b \{p(x)[\dot{u}(x)]^2 + q(x)[u(x)]^2\}\, dx \tag{1}$$

$$H[u] = \int_a^b r(x)[u(x)]^2\, dx \tag{2}$$

where $p(x), r(x) > 0$. We want to find the extreme value of $I[u]$ subject to the constraint $H[u] = 1$. i.e. we want to extremize

$$\begin{aligned} W[u] &= I[u] - \lambda H[u] \\ &= \int_a^b \{p\dot{u}^2 + qu^2 - \lambda r u^2\}\, dx \end{aligned}$$

How does one find the extreme value of a functional? Let us define

$$J[u] = \int_a^b F(x, u, \dot{u})\, dx$$

with $u(a) = c_1, u(b) = c_2$. We can form a family of functions

$$U(x) = u(x) + \epsilon\eta(x).$$

The requirements are $\eta(a) = \eta(b) = 0$ and $u(x)$ is the function which extremizes $J[u]$. We can then replace $u, \dot{u}$ by $U, \dot{U}$ in $J[u]$ and get

$$J(\epsilon) = \int_a^b F(x, U, \dot{U})\, dx$$

where $\dot{U} = \dot{u} + \epsilon\frac{d\eta}{dx}$. Notice that $J(\epsilon)$ is minimum with respect to ϵ when $\epsilon = 0$ regardless of our choice of $\eta(x)$. In other words, $J(\epsilon)$ is a function of ϵ and $\frac{dJ(0)}{d\epsilon} = 0$.

$$\begin{aligned} \frac{dJ(0)}{d\epsilon} &= \int_a^b \{\frac{\partial F}{\partial U}\frac{\partial U}{\partial \epsilon} + \frac{\partial F}{\partial \dot{U}}\frac{\partial \dot{U}}{\partial \epsilon}\}\, dx \\ &= \int_a^b \{\frac{\partial F}{\partial U}\eta + \frac{\partial F}{\partial \dot{U}}\dot{\eta}\}\, dx \\ &= \frac{\partial F}{\partial \dot{U}}\eta|_a^b + \int_a^b [\frac{\partial F}{\partial U} - \frac{d}{dx}\frac{\partial F}{\partial \dot{U}}]\eta\, dx \end{aligned}$$

Since $\eta(a) = \eta(b) = 0$, the first term in the above equation vanishes. Therefore, since $\eta(x)$ is arbitrary, the second term will vanish only if

$$\frac{\partial F}{\partial U} - \frac{d}{dx}(\frac{\partial F}{\partial \dot{U}}) = 0. \tag{3}$$

This is the *Euler-Lagrange Equation.* Apply this to $W[u]$ and we get

$$\frac{d}{dx}(p\dot{u}) - qu + \lambda r u = 0 \tag{4}$$

subject to any set of boundary conditions for which

$$p(x)\dot{u}\eta|_a^b = 0. \tag{5}$$

For example, if $u(a), u(b)$ are specified then η must satisfies the condition that $\eta(a) = \eta(b) = 0$ On the other hand, if $\eta(a), \eta(b)$ are not specified, then we could have $\dot{u}(a) = \dot{u}(b) = 0$. Equation(4) with the appropreiate BCs is called a Sturm-Liouville system and the problem of solving this system is called a Sturm-Liouville eigenvalue problem.

B. Properties of Sturm-Liouville Eigenvalues and Eigenfuctions

1. Sturm-Liouville Theorem: A regular Sturm-Liouville system has an infinite sequence of real eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ with $\lim_{n\to\infty} \lambda_n \to \infty$ and there is one and only one linearly independent eigenfunction $u_n(x)$ corresponds to each eigenvalue.

2. Orthogonality: The eigenfuctions of a Sturm-Liouville system satisfy the condition that $\int_a^b \{r(x)u_j(x)u_k(x)\, dx = \delta_{jk}$.

3. Minimum Principle: If $\hat{u}$ minimizes $W[u]$, then $\hat{u}$ is a eigenfunction of eqn(4) and the corresponding eigenvalue is given by $\lambda = I[\hat{u}]$. Furthermore, given the first k eigenfuctions $u_1, u_2, \ldots, u_k$ with the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, if $\hat{u}$ minimizes $W[u]$ and $\int_a^b \{r(x)\hat{u}(x)u_j(x)\}\, dx = 0, j = 1, 2, \ldots, k$, then $\hat{u}$ is the $(k+1)^{th}$ eigenfunction of eqn(4) and $I[\hat{u}] = \lambda_{k+1} > \lambda_k$ is the $(k+1)^{th}$ eigenvalue.

Proof of the minimum principle: Let $u = \sum_{j=1}^{\infty} c_j u_j$, where u_j are the eigenfuctions of eqn(4) and c_j are real constants. Now

$$\begin{aligned} I[u] &= \int_a^b \{p\dot{u}^2 + qu^2\}\,dx \\ &= \int_a^b \{-\frac{d}{dx}(p\dot{u}) + qu\}u\,dx + p\dot{u}u|_a^b \\ &= \sum_{j=1}^{\infty}\{c_j\lambda_j \int_a^b (ru_j)u\,dx\} \text{ (via eqn(4) and substitute } u \text{ from above)} \\ &= \sum_{j=1}^{\infty} c_j^2\lambda_j \end{aligned}$$

Note that for u to satisfy the orthogonality condition, $\sum_{j=1}^{\infty} c_j^2 = 1$. Thus we could write $(I - \lambda_1)$ as

$$\begin{aligned} I - \lambda_1 &= \sum_{j=2}^{\infty}(\lambda_j - \lambda_1)c_j^2 \\ &\geq 0 \end{aligned}$$

with equality holds when $c_j^2 = 0, j = 2, 3, \ldots$ (since $\lambda_j - \lambda_1 > 0, j = 2, 3, \ldots$). Therefore the first eigenfuction of eqn(4) is $u = u_1$ and the corresponding eigenvalues is $\lambda_1 = I[u_1]$. Furthermore, the next eigenfuction has to satisfy the orthogonality condition so it must be of form $u = \sum_{j=2}^{\infty} c_j u_j$ with $\sum_{j=2}^{\infty} c_j^2 = 1$. We then repeat the above process and find the next eigenfunction to be $u = u_2$ with $\lambda_2 = I[u_2]$. Finally, by repeated go through the above manipulation, we could easily see that the $(k+1)^{th}$ eigenfunction is $u = u_{k+1}$ with $\lambda_{k+1} = I[u_{k+1}]$.

C. An Example

We want to solve the following Sturm-Liouville eigenvalue problem

$$\frac{d}{d\theta}[\sin(\theta)\frac{du}{d\theta}] - \frac{m^2}{\sin(\theta)}u + \lambda\sin(\theta)u = 0,\ \theta \in [0, \pi],\ m \text{ is an integer.} \tag{6}$$

Here, $p(\theta) = r(\theta) = \sin(\theta)$ and $q(\theta) = \frac{m^2}{\sin(\theta)}$. We use a trial function of the form $u_n = \sin(\theta)[\sum_{j=0}^{n} a_j\cos(j\theta) + \sum_{j=1}^{n} b_j\sin(j\theta)$, with a_j, b_j real constants to find the eigenfunctions.

This problem is equivalent to finding u_n that minimumize $W[u] = I[u] - \lambda H[u]$ where

$$\begin{aligned} I[u] &= \int_0^{\pi} \{\sin(\theta)[\sum_{j=0}^{n} a_j(-j\sin(\theta)\sin(j\theta) + \cos(\theta)\cos(j\theta)) \\ &\quad + \sum_{j=1}^{n} b_j(j\sin(\theta)\cos(j\theta) + \cos(\theta)\sin(j\theta))]^2 \\ &\quad + m^2\sin(\theta)[\sum_{j=0}^{n} a_j\cos(j\theta) + \sum_{j=1}^{n} b_j\sin(j\theta)]^2\}\, d\theta \qquad (7) \\ H[u] &= \int_0^{\pi} \sin^3(\theta)[\sum_{j=0}^{n} a_j\cos(j\theta) + \sum_{j=1}^{n} b_j\sin(j\theta)]^2\, d\theta \qquad (8) \end{aligned}$$

This, in turn, is equivalent to solving the system of $(2n+1)$ linear equations

$$\begin{aligned} \frac{\partial I}{\partial a_j} - \lambda\frac{\partial H}{\partial a_j} &= 0 \\ \frac{\partial I}{\partial b_j} - \lambda\frac{\partial H}{\partial b_j} &= 0 \qquad (9) \end{aligned}$$

in (a_j, b_j). From quantum mechanics, we know that the eigenvalues of eqn(6) is of form $\lambda = l(l+1), l = 1, 2, \ldots$, with the corresponding eigenfuctions related to the associated Legendre polynomial. We can show that by solving eqn(9), we recover the the first n eigenvalues and eigenfunctions of eqn(6), provided that $-l \leq m \leq l$. (see the Maple script).

D. Conclusions

We show how, by solving a variational calculus problem of minimizing an certain functional, is equivalent to solving a Sturm-Liouville eigenvalue problem. We also show by using an example how the variational calculus problem associated with a Sturm-Liouville system can be done by solving a system of linear equations.

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[1] Robert Weinstock, *Calculus of Variations With Applications to Physics and Engineering*, McGraw-Hill Book Company Inc., 1952.

[2] James.P. Keener, *Principles of Applied Mathematics: Transformation and Approximation*, HarperCollins Canada, 2000.