Investigations #03 • APMA 935 • Greens Function for the Wave Equation

- Final write-up due, by Noon Friday 17 February. Please submit a progress report to webct by Sunday 12 January.

A) Causality Done Naturally (3 pages) Rederive the Greens function for the forced wave equation in one space dimension

$$G_{tt} - c^2\, G_{xx} = \delta(x - \bar{x})\, \delta(t - \bar{t}) \tag{1}$$

where there are no waves prior to the action of the delta functions, and no incoming waves from $x = \pm\infty$.

As your initial point of departure for the analysism, take a Fourier transform in time. The wave equation becomes an ODE for $\tilde{G}(x, \omega)$ whose solution (for $t > \bar{t}$) have separate expressions for $-\infty < x < \bar{x}$ and $\bar{x} < x < +\infty$. It turns out that there is a unique ODE solution, $\tilde{G}(x, \omega)$, such that the inverse transform has the property that the corresponding $G(x, t)$ is composed of a superposition of only left-going waves for $-\infty < x < \bar{x}$, and only right-going waves for $\bar{x} < x < +\infty$. The rest of the Fourier inversion analysis follows from familiar pole and contour closure arguments.

B) Radiation in a Half-Space (5 pages) Problem #3.8 (Billingham/King, page 79) examines the axisymmetric radiation pattern of a harmonic boundary source. The method of separation of variables proceeds straightforwardly to an integral representation of the solution. As part of your presentation, include the textbook reference you use for the Hankel transform, which is required to determine the Bessel coefficient.

(i) Before embarking on the approximation step, numerically compute a few far-field ($z \gg a$) radiation profiles ($|\phi(r, z)|^2$, for fixed z) to develop an understanding for the radiation pattern. Include examples from both of the suggested limits involving $a\omega/c$.

(ii) The approximation step in the text seems to suggest replacing the Bessel function $J_0(\cdot)$ by an integral representation. Consider the far-field limit as a fixed value of $z \gg a$ with $r = z \tan\alpha$. Interchanging the order of integration would seem to lead to a stationary phase approximation for large z. Past this point, it is not clear how to proceed with the remaining integral over the angle θ.