On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation

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Abstract

We present a formal asymptotic analysis which suggests a model for three-phase boundary motion as a singular limit of a vector-valued Ginzburg-Landau equation. We prove short-time existence and uniqueness of solutions for this model, that is, for a system of three-phase boundaries undergoing curvature motion with assigned angle conditions at the meeting point. Such models pertain to grain boundary motion in alloys. The method we use, based on linearization about the initial conditions, applies to a wide class of parabolic systems. We illustrate this further by its application to an eutectic solidification problem.

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1 Introduction

In this paper, we study some models for three-phase boundary motion. We show formally how a geometrical model of interface motion arises as the singular limit of a vector-valued reaction-diffusion equation. Then, we prove local existence and uniqueness of solutions for the limiting problem using a widely applicable method based on linearization about the initial configuration.

First we study formally the asymptotic behavior as $\varepsilon \to 0$ of the vector-valued Ginzburg-Landau problem

$$u_t = 2\varepsilon^2 \Delta u - \nabla_u W(u) \tag{1}$$

$$\frac{\partial}{\partial n} u|_{\partial\Omega}, = 0 \qquad \text{or} \qquad u(x,t)|_{\partial\Omega} = h(x) \tag{2}$$

$$u(x,0) = g(x), \tag{3}$$

where $u : \Omega \times \mathbf{R}^+ \to \mathbf{R}^m$, with $\Omega \subset \mathbf{R}^n$ and $n \geq 2$, $m \geq 2$. The potential $W : \mathbf{R}^m \to \mathbf{R}$ is non negative and its minimum value zero is attained at three vectors a, b and c, so as to model a three-phase physical system or three grain boundaries meeting along $n-2$ dimensional surfaces. The parameter ε represents the thickness of the transition layer, and is assumed to be small.

The study of (1)–(3) is partially motivated by the work of Allen and Cahn [AC] on the motion of curved antiphase boundaries. Consideration of the gradient flow associated to a free energy functional, modified so as to account for thermodynamic properties of non-uniform systems ([CH],[AC]), lead them to the study of *scalar* Ginzburg-Landau type diffusion equations ([GSS]) of the form of (1),

$$u_t = M\Delta u - \alpha W'(u), \tag{4}$$

where W is an even function with exactly two local minima. In equation (4), α is a positive kinetic coefficient and the diffusion coefficient satisfies $M = 2\alpha K$ where K is the gradient energy coefficient and is proportional to the *square* of the antiphase boundary thickness, which is assumed to be much smaller than the boundary's curvature. Allen and Cahn used (4) to propose that the correct law of motion for antiphase boundaries is mean curvature motion and, in particular, that it is independent of the surface tension of the interface. More precisely the "surface tension" (i.e. the surface energy density) is proportional to $\alpha\sqrt{K}c_0$

where $c_0 = \int_\gamma^\beta \sqrt{W(u)}\, du$, and the constant γ and β are the two local minima of W. However the velocity of the interface is proportional to αK times its mean curvature, and in particular it is *independent* of c_0.

Ginzburg-Landau models such as (1) have been used for many years in the physics and materials literature; see e.g. [AC] for the case of a scalar order parameter, and [BLT] for the vector-valued case. Often they are presented as phenomenological models. But they also have a basis in statistical physics, see e.g. [DeMP]. The theory is now rather complete for the scalar case, in view of the recent papers [Bo] and [KS]. We hope the present study might facilitate similar progress for problems involving several phases, i.e. vector-valued order parameters.

For the system (1)–(3) our formal asymptotic analysis suggests that $u = u^\varepsilon$ separates Ω into several regions where $u^\varepsilon \approx a, b$ or c respectively, and each interface separating these regions moves in the slow time scale $\sigma = \varepsilon^2 t$ with normal velocity equal (as $\varepsilon \to 0$) to the sum of its principal curvatures. To obtain this result we follow the general method of [RSK] and [ORS]. In the process of developing the formal analysis, one needs an explicit representation of standing waves connecting the local minima of W. It turns out that these standing waves come out as a byproduct of works by Sternberg [S] on Γ-convergence for the energy associated to (1) in which W has two zeroes. In fact, one can obtain an explicit representation of the standing waves in terms of geodesics for an appropriate metric, weighted by the potential W (see Lemma 1). There are several results on existence of traveling waves for gradient systems ([R], [Te]) but an added difficulty here is that the wells of W have the same height (and hence the speed is zero). To our knowledge the existence of a standing wave for (1) has not been proven elsewhere.

We then specialize to the case $n = 2$ and derive, besides the motion by curvature for the interfaces, the formula

$$\frac{\sin(\theta_1)}{\Phi^{ca}} = \frac{\sin(\theta_2)}{\Phi^{ab}} = \frac{\sin(\theta_3)}{\Phi^{bc}}, \tag{5}$$

which prescribes the angles between three interfaces at a "triple junction" in terms of the minimum energies $\Phi^{\gamma\beta}$ it takes to change from phase γ to phase β, with $\gamma, \beta = a, b$ or c (see Lemma 1 and Figure 1). Formula (5) is well-known by material scientists working in the theory of phase transitions (e.g. in grain/phase boundary motion [Mu1], [Mu2], [Sm], or in simple fluid phases in equilibrium [W], [C]). Note that, in the particular case where W is symmetric, this formula suggests that the angles between the interfaces must be 120°, as is expected for grain boundaries in an isotropic material. The analysis is further supported by the results of Baldo ([Ba]) on the Γ-convergence problem for the energy associated to (1), which state that this energy converges to a weighted perimeter energy functional whose gradient flow is mean curvature motion.

Our analysis provides further evidence for the correctness of the laws of motion that have been derived by material scientists for the study of three-phase boundary motion (see e.g. [Mu1], [Sm]). Our derivation of the motion law is formal, not rigorous, but we note that the corresponding calculation for the scalar case has been justified rigorously ([BK], [DeMS], [ESS]). We also presume that alternative justifications for the laws of motion could be obtained by other means. For example using arguments in thermomechanics based on balance of forces and moments along the lines of [AG] (see also [FG]).

In the second part of this paper, we present a simple and general method to obtain short-time existence (and uniqueness) of solutions for the three-phase boundary motion model derived in our formal analysis when $n = 2$. More precisely, we prove local existence of smooth solutions for the problem of three curves lying in a domain $\Omega \subset \mathbf{R}^2$, each moving by curvature, which meet at a point with prescribed angle conditions. The other endpoint of each curve meets the boundary $\partial\Omega$, also with a prescribed angle. In particular, this answers a question posed by Mullins about the well-posedness of this problem ([Mu1]). The method we use is based upon linearizing the problem about the initial data and verifying that the linearized boundary conditions satisfy the "complementary condition" for the resulting parabolic system (see e.g. [So]). This condition assures the existence of a solution to the linearized problem which is then used to establish local existence for the full problem via a fixed point argument. This method easily extends to the case of networks in which there are many "triple junctions", and to situations where the physical system exhibits any number of phases. It also applies to some cases of anisotropic mean curvature motion when the problem is a non-degenerate second order parabolic problem. To show further the wide applicability of the method, we sketch the proof of local existence of solutions for another problem in the theory of phase transitions, namely eutectic solidification ([K],[W]). We study a simplified model for lamellar eutectics. In this model, two curves (the solid-liquid interfaces) move normally with a speed whose dominant contribution is proportional to their curvature. As the two curves evolve, the locus of their meeting point traces out a third curve (the solid-solid interface) that should maintain a fixed angle with the solid-liquid interfaces at the meeting point (see Figure 4).

We emphasize that our existence theory is logically separate from our derivation of the laws of motion. It is fully rigorous but local-in-time. The problem is that when a geometric singularity occurs, for example when two triple points coalesce, our parametrization of the problem becomes singular. To get a global-in-time existence theory would require a different approach, based on some kind of weak solution. In the scalar case the "level-set formulation" and the theory of viscosity solutions has been used for this purpose with striking success ([CGG], [ES]). So far there is no indication of how to extend that work to problems involving three or more phases. However see [T] for related ideas.

Several discrete models have been suggested to simulate the evolution of grain growth. Among these we can cite the Potts model (see [GAG] and references therein), vertex and boundary dynamics models (see e.g. [CN] and [KNN]), mean-field theories ([FSU]) and motion by crystalline curvature ([T]). (See also [CHT] for a survey of several approaches to defining and computing geometric motion of interfaces.) In this regard, we believe that the model we study in this paper, which is certainly amenable to discretizations, is very likely to yield valuable numerical results. Research in this direction will be left for further work.

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2 Formal Analysis

In this section, we use a multiple time scales asymptotic analysis to obtain formally the asymptotic behavior of the solution $u = u^\varepsilon$ of (1)–(3) as $\varepsilon \to 0$. Specifically, we show that u^ε divides Ω into regions where $u^\varepsilon \approx a$, b or c and that the interfaces dividing these regions evolve normally with speed equal to their mean curvature. In the two-dimensional case, we also derive a formula for the angles between the interfaces at their meeting point in terms of a metric involving the potential W and the equilibria a, b and c.

We shall follow closely two papers in which the scalar version of (1)–(3) is studied: the first one is the paper of Rubinstein - Sternberg - Keller [RSK] where the boundary conditions are of Neumann type, and the second is the paper of Owen - Rubinstein - Sternberg [ORS] where the Dirichlet problem is studied. In [RSK] it is formally shown that the interfaces meet the boundary of Ω with a $90°$ angle, while the results in [ORS] suggest that the contact angle depends on the boundary data h and on the potential W. The essential difference between our formal analysis and that of [RSK] and [ORS] is the behavior of u^ε near triple junctions. Hence we shall modify ideas from [ORS] to study the behavior near triple juntions and mostly quote results of [RSK] and [ORS] for the study of u^ε in the interior of Ω and near the boundary $\partial\Omega$.

First we seek the "outer expansion" in the original time scale. For this, we write u^ε in the form of

$$u^\varepsilon = u_0^{out}(x,\, t) + \varepsilon^2 u_1^{out}(x,\, t) + \varepsilon^4 u_2^{out}(x,\, t) + ... \qquad (\varepsilon << 1). \tag{6}$$

Substituting (6) in (1) and (3), we obtain

$$\frac{\partial}{\partial t} u_0^{out} = -\nabla_u W(u_0^{out}) \tag{7}$$

$$u_0^{out}(x,\, 0) = g(x). \tag{8}$$

Since W is a non-negative potential with minima at a, b and c, it follows from (7) that $u_0^{out}(x,t)$ tends to a, b or c as $t \to \infty$. For simplicity, we shall assume that $\lim_{t\to\infty} u_0^{out}$ divides Ω into 3 regions, and we let Γ_1 divide phase a and phase b, Γ_2 divide phase b and phase c and finally we let Γ_3 divide phase c and phase a (see Figure 1). Hence interfaces generate in this time scale. Subsequently, the diffusion term in (1) becomes large and we anticipate that these interfaces will evolve. Therefore a better approximation for u^ε near the transition layers is obtained by expanding it in a slower time scale. Moreover, we expect a different asymptotic behavior near $\partial\Omega$ and near the triple junction. Therefore we must study the solution u^ε separately in three regions: near the interior transition layers, where the interfaces meet $\partial\Omega$ and, finally, around the triple junction.

First, we study the interior transition layer through Γ_i. Following [RSK], we introduce the slow time scale $\sigma = \varepsilon^2\, t$ and rescaled coordinates

$$\delta_i = \varepsilon^{-1} d_i(x,\, \sigma), \tag{9}$$

where $d_i(x,\, \sigma)$ is the signed distance from x to the interface Γ_i. Expanding in these new variables,

$$u^\varepsilon = u_{0i}^{t\ell}(\delta_i,\, \sigma) + \varepsilon \;\; u_{1i}^{t\ell}(\delta_i,\, \sigma) + \varepsilon^2 \;\; u_{2i}^{t\ell}(\delta_i,\, \sigma) + ... \tag{10}$$

Next we substitute (10) in (1) with $i = 1$ and, by carrying the calculation up to second order, it follows ([RSK]):

$$2(u_{01}^{t\ell})_{\delta_1\delta_1} = \nabla_u W(u_{01}^{t\ell}) \tag{11}$$

$$u_{01}^{t\ell} \to a \text{ as } \quad \delta_1 \to -\infty, \qquad u_{01}^{t\ell} \to b \text{ as } \quad \delta_1 \to \infty. \tag{12}$$

$$\frac{\partial}{\partial\sigma} d_1 = k_{d_1}\,, \tag{13}$$

where k_{d_1} is the mean curvature of the level sets of d_1. Thus the interface Γ_1 evolves in the slow time scale σ according to its mean curvature and the profile of the transition layer is a standing wave given by the solution to (11) and (12). A similar calculation can be carried out for the other two interfaces Γ_2 and Γ_3 with the same conclusion except that (12) is replaced by (see Figure 1)

$$u_{02}^{t\ell} \to b \text{ as } \delta_2 \to -\infty, \qquad u_{02}^{t\ell} \to c \text{ as } \delta_2 \to \infty,$$

$$u_{03}^{t\ell} \to c \text{ as } \delta_3 \to -\infty, \qquad u_{03}^{t\ell} \to a \text{ as } \delta_3 \to \infty.$$

Here we make the observation that standing wave solutions arise as a byproduct of work by Sternberg [S] on Γ-convergence for the energy associated to (1) in the case that W vanishes on 2 vectors. It turns out that the geodesics connecting the local minima of W obtained using a weighted "distance" functional yield an exact representation for the standing waves.

Lemma 1 *Let*

$$\Phi^{ab}(b) = \inf_{\substack{p\in C^1 \\ p(-1)=a,\, p(1)=b}} 2\int_{-1}^{1} \sqrt{W(p(t))}|p'(t)|dt, \tag{14}$$

and let $p_{ab}(t)$ be a geodesic connecting a to b, i.e. a path p which achieves the infimum in (14). Then there exists a smooth increasing function $\beta : (-\infty,\infty) \to (-1,1)$ such that the curve $\gamma_{ab}(\delta_1) \equiv p_{ab}(\beta(\delta_1))$ is a solution to (11)-(12). In particular, given a unit vector e, $u^\varepsilon(x,t) \equiv \gamma_{ab}\left(\frac{x\cdot e}{\varepsilon}\right)$ is a solution to (1) connecting a to b. The function γ_{ab} satisfies

$$|\partial_{\delta_1}\gamma_{ab}|^2 = W(\gamma_{ab}),$$

and

$$\Phi^{ab}(b) = 2\int_{-\infty}^{\infty} W(\gamma_{ab}(\delta_1))d\delta_1\,.$$

This Lemma follows easily from the proof of the Lemma in [S].

Regarding the behavior of u^ε near $\partial\Omega$, we just recall that in [RSK] it is shown that if the boundary conditions are of Neumann type, Γ_i meets $\partial\Omega$ at a 90° angle. In the case of Dirichlet boundary conditions, $u^\varepsilon(x,\, t) = h(x)$ on $\partial\Omega$, results of [ORS] show that Γ_i must meet $\partial\Omega$ at a prescribed (non-zero) angle which depends only on h and on the potential W.

Finally, we study the behavior of u^ε near the triple junction of the interfaces. To simplify the argument, we assume from now on that $n = 2$, i.e. $u^\varepsilon(\cdot, t) : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^m$. Recalling that the interior interfaces evolve in the slow time scale $\sigma = \varepsilon^2 t$, we let $m(\sigma)$ and θ_i, $i = 1, 2$, denote the meeting point and the angles between the tangent planes to Γ_i and to Γ_{i+1} at $m(\sigma)$. Then the angle between Γ_1 and Γ_3 satisfies $\theta_3 = 2\pi - \theta_1 - \theta_2$ (see Figure 1). In order to determine the angles between the interfaces at $m(\sigma)$, we introduce the following systems of stretched coordinates around $m(\sigma)$

$$\eta_i = \frac{x - m(\sigma)}{\varepsilon},$$

with $\eta_i = (\xi_i, \zeta_i)$ where ζ_i represents coordinates along the tangent plane to Γ_i at $m(\sigma)$ and ξ_i is perpendicular to ζ_i (see Figure 2). Next, let T be an isosceles triangle with base perpendicular to Γ_1 as in Figure 2. Since the interior interfaces evolve in the slow time scale σ, we expand u^ε near $\eta_1 = 0$ in the form

$$u^\varepsilon = u_0^{in}(\eta_1,\, \sigma) + \varepsilon\; u_1^{in}(\eta_1,\, \sigma) + ...$$

Using (1), u_0^{in} satisfies

$$2\Delta_{\eta_1} u_0^{in} = \nabla_u W(u_0^{in}). \tag{15}$$

Furthermore, we have the following matching conditions between u_0^{in} and $u_{0i}^{t\ell}$ ([VD]):

$$\lim_{\zeta_1 \to \infty} u_0^{in}(\xi_1,\, \zeta_1) = u_{01}^{t\ell}(\xi_1), \tag{16}$$

$$\lim{}^* u_0^{in}(\xi_1,\, \zeta_1) = u_{0i}^{t\ell}(\xi_i), \tag{17}$$

where $\lim^*$ denotes the limit as $|\zeta_1|$ and $|\xi_1| \to \infty$ along the lines which are at some distance ξ_i from the interface Γ_i (see Figure 3). We note that from (15), (16) and (17), it follows that u_0^{in} is in fact independent of σ. Since the coordinates (ξ_i, ζ_i) satisfy

$$\xi_i = -\sin(\gamma_i)\xi_1 - \cos(\gamma_i)\zeta_1 \quad \text{and} \quad \zeta_i = \cos(\gamma_i)\xi_1 - \sin(\gamma_i)\zeta_1$$

where $0 \leq \gamma_i < 2\pi$ is the angle between Γ_i and the ξ_1-axis, the matching conditions (16)–(17) can be written as

$$\lim_{\zeta_i \to \infty} u_0^{in}(\xi_i,\, \zeta_i) = u_{0i}^{t\ell}(\xi_i) \tag{18}$$

for ξ_i fixed. Following ideas from [ORS], we now multiply (15) by $\partial_{\xi_1} u_0^{in}$ and we integrate over T :

$$\int\int_T \partial_{\xi_1} u_0^{in} \cdot \nabla_u W(u_0^{in})\, d\eta_1 = \int\int_T 2\partial_{\xi_1} u_0^{in} \cdot \partial_{\xi_1}^2 u_0^{in} + 2\partial_{\xi_1} u_0^{in} \cdot \partial_{\zeta_1}^2 u_0^{in}\, d\eta_1.$$

Since this can be written as

$$\int\int_T \partial_{\xi_1}[W(u_0^{in}) + |\partial_{\zeta_1} u_0^{in}|^2 - |\partial_{\xi_1} u_0^{in}|^2]\, d\eta_1 = \int\int_T 2\partial_{\zeta_1}(\partial_{\xi_1} u_0^{in} \cdot \partial_{\zeta_1} u_0^{in})\, d\eta_1\,,$$

using the divergence theorem it follows that

$$\int_{\partial T} \left[W(u_0^{in}) + |\partial_{\zeta_1} u_0^{in}|^2 - |\partial_{\xi_1} u_0^{in}|^2\right] \nu_1 ds = -\int_{\partial T} 2\partial_{\xi_1} u_0^{in} \cdot \partial_{\zeta_1} u_0^{in}\, \nu_2 ds, \tag{19}$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal vector to ∂T.

Next, we parametrize these line integrals in the $(\xi_i,\, \zeta_i)$ coordinates. The last two summands of the integrand in the left hand side of (19) become

$$|\partial_{\zeta_1} u_0^{in}|^2 - |\partial_{\xi_1} u_0^{in}|^2 = -\cos(2\gamma_i)|\partial_{\zeta_i} u_0^{in}|^2 + 4\partial_{\zeta_i} u_0^{in} \cdot \partial_{\xi_i} u_0^{in} \sin(\gamma_i)\cos(\gamma_i) + |\partial_{\xi_i} u_0^{in}|^2 \cos(2\gamma_i)$$

while the integrand in the right hand side of (19) satisfies

$$-\partial_{\xi_1} u_0^{in} \cdot \partial_{\zeta_1} u_0^{in} = |\partial_{\zeta_i} u_0^{in}|^2 \cos(\gamma_i)\sin(\gamma_i) + \cos(2\gamma_i)\partial_{\zeta_i} u_0^{in} \cdot \partial_{\xi_i} u_0^{in} - |\partial_{\xi_i} u_0^{in}|^2 \cos(\gamma_i)\sin(\gamma_i).$$

Assuming that $0 < \theta_i < \pi$, $i = 1, 2, 3$, and choosing counterclockwise orientation along ∂T, we can write $d\xi_1 = ds$, $d\xi_2 = \sin(\gamma_2 + \theta)ds$ and $d\xi_3 = \sin(\gamma_3 - \theta)ds$, where θ is the angle at the base of T (see Figure 2). Now using this change of variables, the matching conditions (18) and the fact that $\lim_{|\zeta_i| \to \infty} |\partial_{\zeta_i} u_0^{in}| = 0$ for finite energy solutions (this also follows from the matching conditions since $u_{0i}^{t\ell}$ is independent of ζ_i), we take the limit as the length R of the base of T tends to ∞. Since we want to use the matching conditions (18), we need to consider two domains of integration within each line integral. The first domain lies within a fixed distance α to Γ_i while the second consists of those points which are at least α-away from Γ_i. In taking the limit as $R \to \infty$ and then as $\alpha \to \infty$, the line integrals which are away from Γ_i tend to zero: indeed the integrands tend to zero since $u_0^{in} \to u_0^{out}$ which, in this time scale, is either a, b or c. Therefore taking the limit in this order, we obtain on the left hand side of (19)

$$\lim_{\alpha\to\infty} \lim_{\substack{R\to\infty \\ |\xi_i|<\alpha}} \int_{\partial T} \left(W(u_0^{in}) + |\partial_{\zeta_1} u_0^{in}|^2 - |\partial_{\xi_1} u_0^{in}|^2 \right) \nu_1 ds =$$

$$\int_{-\infty}^{\infty} \left[|\partial_{\xi_2} u_{02}^{t\ell}|^2 \cos(2\gamma_2) + W(u_{02}^{t\ell}) \right] \frac{\sin(\theta)}{\sin(\gamma_2+\theta)} d\xi_2$$

$$- \int_{-\infty}^{\infty} \left[|\partial_{\xi_3} u_{03}^{t\ell}|^2 \cos(2\gamma_3) + W(u_{03}^{t\ell}) \right] \frac{\sin(\theta)}{\sin(\gamma_3-\theta)} d\xi_3 \,,$$

where we used that $\nu_1 \, d\xi_1 = 0$ and the fact that in the R limit $\zeta_i >> 1$ so that we can use the matching condition (18). Similarly, we obtain for the right hand side of (19)

$$\lim_{\alpha\to\infty} \lim_{\substack{R\to\infty \\ |\xi_i|<\alpha}} - \int_{\partial T} 2 \; \partial_{\xi_1} u_0^{in} \cdot \partial_{\zeta_1} u_0^{in} \, \nu_2 ds =$$

$$-2 \int_{-\infty}^{\infty} |\partial_{\xi_2} u_{02}^{t\ell}|^2 \cos(\gamma_2) \sin(\gamma_2) \frac{\cos(\theta)}{\sin(\gamma_2+\theta)} d\xi_2$$

$$-2 \int_{-\infty}^{\infty} |\partial_{\xi_3} u_{03}^{t\ell}|^2 \cos(\gamma_3) \sin(\gamma_3) \frac{\cos(\theta)}{\sin(\gamma_3-\theta)} d\xi_3 \,,$$

where this time the integral along $d\xi_1$ vanishes in the limit since $\gamma_1 = 270°$.

In view of (19), it follows

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[|\partial_{\xi_2} u_{02}^{t\ell}|^2 \cos(2\gamma_2) + W(u_{02}^{t\ell}) \right] \frac{\sin(\theta)}{\sin(\gamma_2+\theta)} d\xi_2 \\ &- \int_{-\infty}^{\infty} \left[|\partial_{\xi_3} u_{03}^{t\ell}|^2 \cos(2\gamma_3) + W(u_{03}^{t\ell}) \right] \frac{\sin(\theta)}{\sin(\gamma_3-\theta)} d\xi_3 \\ &= -2 \int_{-\infty}^{\infty} |\partial_{\xi_2} u_{02}^{t\ell}|^2 \cos(\gamma_2) \sin(\gamma_2) \frac{\cos(\theta)}{\sin(\gamma_2+\theta)} d\xi_2 \\ &-2 \int_{-\infty}^{\infty} |\partial_{\xi_3} u_{03}^{t\ell}|^2 \cos(\gamma_3) \sin(\gamma_3) \frac{\cos(\theta)}{\sin(\gamma_3-\theta)} d\xi_3 \,. \end{aligned} \tag{20}$$

Next we obtain the angle conditions at the meeting point. Indeed applying Lemma 1 to (20) yields

$$\cos^2(\gamma_2)\frac{\sin(\theta)}{\sin(\gamma_2+\theta)}\Phi^{bc}(c) - \cos^2(\gamma_3)\frac{\sin(\theta)}{\sin(\gamma_3-\theta)}\Phi^{ca}(a) =$$

$$-\cos(\gamma_2)\sin(\gamma_2)\frac{\cos(\theta)}{\sin(\gamma_2+\theta)}\Phi^{bc}(c) - \cos(\gamma_3)\sin(\gamma_3)\frac{\cos(\theta)}{\sin(\gamma_3-\theta)}\Phi^{ca}(a)$$

which after simplification, gives

$$\cos(\gamma_2)\,\Phi^{bc}(c) = -\cos(\gamma_3)\,\Phi^{ca}(a).$$

Further, since $\gamma_2 + \frac{\pi}{2} = \theta_1$ and $\gamma_3 - \gamma_2 = \theta_2$, it follows that $\gamma_3 = \frac{3\pi}{2} - \theta_3$ and we conclude

$$\sin(\theta_1)\,\Phi^{bc}(c) = \sin(\theta_3)\,\Phi^{ca}(a). \tag{21}$$

If we rotate T so that its base is around Γ_2, we can similarly derive the equation

$$\sin(\theta_2)\,\Phi^{ca}(a) = \sin(\theta_1)\,\Phi^{ab}(b). \tag{22}$$

Therefore (21) and (22) give the angle conditions between the three interfaces. We note that (21) and (22) are the well-known formulae for the angles between three interfaces in grain/phase boundary motion ([Mu1], [Mu2], [Sm]), or in simple fluid phases in equilibrium ([W], [C]), where $\Phi^{ab}(b)$ represents surface tension of the interface between phase a and b. Also note that if W is symmetric $\Phi^{ca}(a) = \Phi^{ab}(b) = \Phi^{bc}(c)$ and (21)–(22) imply that the interfaces must meet at an angle of $120°$.

The case in which one of the angles $\theta_i \geq \pi$ while the others are positive, cannot occur: indeed one can easily check that (21) and (22) would continue to hold, which is impossible since $\Phi^{\beta\gamma}(\gamma) > 0$ for β and γ equal to either a, b or c. On the other hand, if one of the angles vanishes the limit problem becomes ill-posed (see Remark 2).

In conclusion, it follows from (13) that the interfaces Γ_i, $i = 1,2,3$, evolve normally according to their mean curvature and the angles they form at triple junctions are given by (21) and (22). In the two dimensional case, parametrizing Γ_i, $i = 1,2,3$, in arc-length coordinates, we have shown formally that the solution u^ε of (1)–(3) asymptotically yields the following three-phase boundary problem for the curves Γ_i:

$$\begin{aligned}
&\Gamma_{it} = \Gamma_{iss} \qquad \text{in } D_t \equiv \{\,(s,t)\,/\,0 \le s \le L_i(t)\,\} \qquad (i=1,2,3)\\
&\Gamma_i(s,0) = \Gamma_i^0(s)\\
&\frac{\Gamma_{1s}}{|\Gamma_{1s}|}\cdot\frac{\Gamma_{2s}}{|\Gamma_{2s}|} = \cos(\theta_1) \qquad \text{at } s=0\\
&\frac{\Gamma_{2s}}{|\Gamma_{2s}|}\cdot\frac{\Gamma_{3s}}{|\Gamma_{3s}|} = \cos(\theta_2) \qquad \text{at } s=0\\
&\Gamma_1(0,t) = \Gamma_2(0,t) = \Gamma_3(0,t)\\
&\Gamma_i \perp \partial\Omega \qquad \text{at } s = L_i(t), \text{ in the case of Neumann boundary conditions,}\\
&\angle(\Gamma_i,\,\partial\Omega) = \alpha_i \qquad \text{at } s = L_i(t), \text{ in the case of Dirichlet type boundary conditions,}
\end{aligned} \tag{23}$$

where $0 < \theta_i < \pi$ and $\alpha_i \neq 0$.

3 Short-time existence

In this section, we present a local existence result for the three-phase boundary problem (23) derived in the previous section. While arclength parametrization is geometrically convenient, it will be easier, from our perspective, to write the equations in coordinates in which the spatial and time variables are independent. In order to find the simplest formulation, let us first suppose that the curves are given by the graphs of functions v^j, $j = 1,2,3$, and let $(\mu(t), v^j(\mu(t),t))$ and $(l_j(t), v^j(l_j(t),t))$ be, respectively, the meeting point of the three curves and the intersection point of each curve with $\partial\Omega$. Then (23) can be written in the form

$$v_t^1 = \frac{v_{yy}^1}{1+(v_y^1)^2} \qquad \mu(t) \le y \le l_1(t)$$

$$v_t^j = \frac{v_{yy}^j}{1+(v_y^j)^2} \qquad l_j(t) \le y \le \mu(t) \quad j = 2,3$$

$$v^1(\mu(t),t) = v^2(\mu(t),t) = v^3(\mu(t),t)$$

$$\frac{(1, v^1_y)}{|(1, v^1_y)|} \cdot \frac{(-1, -v^2_y)}{|(-1, -v^2_y)|} = \cos(\theta_1) \qquad \text{at } y = \mu(t) \tag{24}$$

$$\frac{(1, v^1_y)}{|(1, v^1_y)|} \cdot \frac{(-1, -v^3_y)}{|(-1, -v^3_y)|} = \cos(\theta_2) \qquad \text{at } y = \mu(t)$$

$$\angle\left((1, v^1_y), \partial\Omega\right) = \alpha_1 \quad \text{at } y = l_1(t)$$

$$\angle\left((-1, -v^j_y), \partial\Omega\right) = \alpha_j \quad \text{at } y = l_j(t) \quad j = 2, 3$$

$$b(l_j(t), v^j(l_j(t), t)) = 0, \qquad j = 1, 2, 3$$

where

$$\partial\Omega = \{(x_1, x_2) \in \mathbf{R}^2 \,| b(x_1, x_2) = 0\}.$$

Local existence and uniqueness for this "free boundary problem" can be established directly via a fixed point argument (in fact the proof becomes easier if one considers the system of equations satisfied by $w^j = v^j_y$). However a more straightforward formulation of the problem (23) which does not involve free boundaries can be derived. In fact, if the position vector for Γ_1 is given by $(u_1(x,t), u_2(x,t))$ ($x \in [0,1]$) and if u_1 is invertible, then letting $y = u_1(x,t)$ and $v^1(y,t) = u_2(u_1^{-1}(y,t),t)$, one can easily show from (24) that $p_1 = (u_1, u_2)$ satisfies

$$p_{1t} \cdot N_1 = \frac{p_{1xx}}{|p_{1x}|^2} \cdot N_1 \equiv k_1 \,, \tag{25}$$

where N_1 is the unit normal to Γ_1 and k_1 is its curvature. Since (23) does not prescribe the tangential velocity, a choice must be made, and motivated by (25) it is natural to consider the system

$$p_{jt} = \frac{p_{jxx}}{|p_{jx}|^2} \qquad x \in [0,1]\,, \; j = 1, 2, 3$$

$$p_j(x, 0) = p^0_j(x)$$

$$p_1(0,t) = p_2(0,t) = p_3(0,t)$$

$$\frac{p_{1x}}{|p_{1x}|} \cdot \frac{p_{2x}}{|p_{2x}|} = \cos(\theta_1) \qquad \text{at } x = 0 \tag{26}$$

$$\frac{p_{2x}}{|p_{2x}|} \cdot \frac{p_{3x}}{|p_{3x}|} = \cos(\theta_2) \qquad \text{at } x = 0$$

$$\frac{p_{jx}}{|p_{jx}|} \cdot \frac{\vec{T}(p_j)}{|\vec{T}(p_j)|} = \cos(\alpha_j) \qquad \text{at } x = 1$$

$$b(p_j) = 0 \qquad \text{at } x = 1\,,$$

where

$$\vec{T}(p_j) = (-\partial_{x_2} b(p_j), \partial_{x_1} b(p_j))$$

is tangent to $\partial\Omega$ at $p_j(1,t)$. If we let

$$p_1 = (u_1, u_2), \quad p_2 = (u_3, u_4) \text{ and } p_3 = (u_5, u_6), \tag{27}$$

then (26) is equivalent to the following system of parabolic equations for the u_j's:

$$u_{jt} = \frac{1}{|p_{1x}|^2} u_{jxx} \qquad j = 1, 2$$

$$u_{jt} = \frac{1}{|p_{2x}|^2} u_{jxx} \qquad j = 3, 4$$

$$u_{jt} = \frac{1}{|p_{3x}|^2} u_{jxx} \qquad j = 5, 6$$

$$\begin{aligned}
& u_j(x,0) = u_j^0(x) \\
& u_j(0,t) = u_{j+2}(0,t) = u_{j+4}(0,t) \qquad j = 1, 2 \\
& u_{1x}u_{3x} + u_{2x}u_{4x} - \cos(\theta_1)|p_{1x}||p_{2x}| = 0 \qquad \text{at } x = 0 \\
& u_{3x}u_{5x} + u_{4x}u_{6x} - \cos(\theta_2)|p_{2x}||p_{3x}| = 0 \qquad \text{at } x = 0 \\
& -u_{1x}\partial_{x_2} b(p_1) + u_{2x}\partial_{x_1} b(p_1) - \cos(\alpha_1)|p_{1x}||\vec{T}(p_1)| = 0 \qquad \text{at } x = 1 \\
& -u_{3x}\partial_{x_2} b(p_2) + u_{4x}\partial_{x_1} b(p_2) - \cos(\alpha_2)|p_{2x}||\vec{T}(p_2)| = 0 \qquad \text{at } x = 1 \\
& -u_{5x}\partial_{x_2} b(p_3) + u_{6x}\partial_{x_1} b(p_3) - \cos(\alpha_3)|p_{3x}||\vec{T}(p_3)| = 0 \qquad \text{at } x = 1 \\
& b(p_j) = 0 \qquad \text{at } x = 1 \quad j = 1, 2, 3.
\end{aligned} \tag{28}$$

We shall prove the following theorem:

Theorem 1 *Let* $u_j^0(x) \in C^{2+\alpha}([0,1])$ $(0 < \alpha < 1)$ *satisfy the compatibility conditions for* (28)*, and assume that* $\partial\Omega$ *is* $C^{2+\alpha}$*. Then if* $\alpha_j \neq 0$, $j = 1,2,3$, $0 < \theta_j < \pi$, $j = 1,2$, *and* $\delta = \min_{1\leq j\leq 3} \inf_x |p_{jx}^0(x)| > 0$, *there exists* $T = T(|u_j^0(x)|_{2+\alpha}, |b|_{2+\alpha}, \delta)$ *such that* (28) *has a unique solution in* $C^{2+\alpha,1+\frac{\alpha}{2}}([0,1]\times[0,T])$.

By reparametrizing, it follows:

Corollary 1 *If* $\Gamma_j(s,0) \in C^{2+\alpha}(D_0)$*, and satisfy the boundary conditions in (23), then (23) has a unique solution in* $C^{2+\alpha,1+\frac{\alpha}{2}}(D_T)$.

Remark 1 The theorem also holds if the angles θ_j between the curves, and the angles α_j between the curves and the boundary $\partial\Omega$, change in time as long as $\theta_j(t), \alpha_j(t) \in C^{\frac{1+\alpha}{2}}([0,T])$.

In order to keep the presentation simpler, we choose $\theta_1 = \theta_2 = 120°$ and $\alpha_j = 90°$ in what follows. The modifications necessary to treat the general case are simple and will be pointed out later on. The idea of the proof of the theorem is to first linearize the system (28) about the initial data, use the classical theory for parabolic systems (see e.g. [So]) to establish existence for the linearized system and then obtain local existence for the full problem by means of a fixed point argument. More precisely, if we let

$$X_j \equiv \{u \in C^{2+\alpha,1+\frac{\alpha}{2}}([0,1]\times[0,T]) \mid |u|_{2+\alpha} \leq M \text{ and } u(x,0) = u_j^0(x)\} \quad 1 \leq j \leq 6\,,$$

we seek a fixed point of the map

$$\mathcal{R} : \prod_{j=1}^{6} X_j \to \prod_{j=1}^{6} X_j$$

which associates to $\bar{u}_j \in X_j$, the solution $u = \mathcal{R}\bar{u}$ of the linearized system:

$$u_{jt} - D_j u_{jxx} = f_j \tag{29}$$

$$u_j(x,0) = u_j^0(x) \tag{30}$$

where (cf. (27), (28))

$$D_j = \begin{cases} \frac{1}{|p_{1x}^0|^2} & \text{for } j = 1,2 \\ \frac{1}{|p_{2x}^0|^2} & \text{for } j = 3,4 \\ \frac{1}{|p_{3x}^0|^2} & \text{for } j = 5,6 \end{cases} \quad \text{and } f_j = \begin{cases} \left(\frac{1}{|\bar{p}_{1x}|^2} - \frac{1}{|p_{1x}^0|^2}\right)\bar{u}_{jxx} & \text{for } j = 1,2 \\ \left(\frac{1}{|\bar{p}_{2x}|^2} - \frac{1}{|p_{2x}^0|^2}\right)\bar{u}_{jxx} & \text{for } j = 3,4 \\ \left(\frac{1}{|\bar{p}_{3x}|^2} - \frac{1}{|p_{3x}^0|^2}\right)\bar{u}_{jxx} & \text{for } j = 5,6 \end{cases} \tag{31}$$

with the linearized boundary conditions:

$$B_j(0,t,u) \equiv u_j(0,t) - u_{j+2}(0,t) = 0 \quad 1 \le j \le 4 \tag{32}$$

$$\begin{aligned} B_5(0,t,u) &\equiv \left(u^0_{3x} + \frac{1}{2}u^0_{1x}\frac{|p^0_{2x}|}{|p^0_{1x}|}\right) u_{1x} + \left(u^0_{4x} + \frac{1}{2}u^0_{2x}\frac{|p^0_{2x}|}{|p^0_{1x}|}\right) u_{2x} \\ &\quad + \left(u^0_{1x} + \frac{1}{2}u^0_{3x}\frac{|p^0_{1x}|}{|p^0_{2x}|}\right) u_{3x} + \left(u^0_{2x} + \frac{1}{2}u^0_{4x}\frac{|p^0_{1x}|}{|p^0_{2x}|}\right) u_{4x} \\ &\equiv b_{51}u_{1x} + b_{52}u_{2x} + b_{53}u_{3x} + b_{54}u_{4x} \\ &= \Phi(\bar{p}_1, \bar{p}_2) \end{aligned} \tag{33}$$

$$\begin{aligned} B_6(0,t,u) &\equiv \left(u^0_{5x} + \frac{1}{2}u^0_{3x}\frac{|p^0_{3x}|}{|p^0_{2x}|}\right) u_{3x} + \left(u^0_{6x} + \frac{1}{2}u^0_{4x}\frac{|p^0_{3x}|}{|p^0_{2x}|}\right) u_{4x} \\ &\quad + \left(u^0_{3x} + \frac{1}{2}u^0_{5x}\frac{|p^0_{2x}|}{|p^0_{3x}|}\right) u_{5x} + \left(u^0_{4x} + \frac{1}{2}u^0_{6x}\frac{|p^0_{2x}|}{|p^0_{3x}|}\right) u_{6x} \\ &\equiv b_{63}u_{3x} + b_{64}u_{4x} + b_{65}u_{5x} + b_{66}u_{6x} \\ &= \Psi(\bar{p}_2, \bar{p}_3) \end{aligned} \tag{34}$$

and

$$\begin{aligned} B_j(1,t,u) &\equiv p_{jx} \cdot \vec{T}(p^0_j) = \Xi_j(\bar{p}_j) \qquad j = 1,2,3 \\ B_{j+3}(1,t,u) &\equiv Db(p^0_j) \cdot p_j = \Xi_{j+3}(\bar{p}_j) \quad j = 1,2,3. \end{aligned} \tag{35}$$

The functions Φ, Ψ and Ξ are defined by

$$\Phi(\bar{p}_1, \bar{p}_2) = -\bar{p}_{1x}(0,t) \cdot \bar{p}_{2x}(0,t) - \frac{1}{2}|\bar{p}_{1x}(0,t)||\bar{p}_{2x}(0,t)| + \bar{B}_5(0,t,u) \tag{36}$$

$$\Psi(\bar{p}_2, \bar{p}_3) = -\bar{p}_{2x}(0,t) \cdot \bar{p}_{3x}(0,t) - \frac{1}{2}|\bar{p}_{2x}(0,t)||\bar{p}_{3x}(0,t)| + \bar{B}_6(0,t,u)$$

and

$$\begin{aligned} \Xi_j(\bar{p}_j) &= -\bar{p}_{jx}(1,t) \cdot \vec{T}(\bar{p}_j(1,t)) + \bar{B}_j(1,t,u) \quad j = 1,2,3 \\ \Xi_{j+3}(\bar{p}_j) &= -b(\bar{p}_j(1,t)) + \bar{B}_{j+3}(1,t,u) \qquad j = 1,2,3. \end{aligned} \tag{37}$$

(Here and throughout, an expression with a bar above means that u_j is replaced by $\bar{u}_j$ in the expression. Also to simplify the notation, we shall write $C^{2+\alpha}$ instead of $C^{2+\alpha,1+\frac{\alpha}{2}}([0,1]\times[0,T])$, and $|\cdot|_{2+\alpha}$ for the norm in $C^{2+\alpha}$.)

We shall show that in fact, for a suitable constant M and small enough T, the map $\mathcal{R}$ is contracting. The success of this approach depends solely on the uniform parabolicity of the system (28) (for small time) and the fact that the boundary conditions (32)–(35) are "complementary" for the equations (29) ([So, p. 11]), i.e. that the system (29)–(35) is well-posed.

Proof of Theorem 1: First we show that there exists a solution in $C^{2+\alpha}$ for the problem (29)–(35). This follows from the classical theory of parabolic systems. In fact, (29)–(35) has a unique solution (u_j) satisfying

$$\sum_{j=1}^{6}|u_j|_{2+\alpha}\le C_s\left(\sum_{j=1}^{6}|f_j|_\alpha+\sum_{j=1}^{6}|u_j^0|_{2+\alpha}+|\Phi|_{1+\alpha}+|\Psi|_{1+\alpha}+\sum_{j=1}^{6}|\Xi_j|_{1+\alpha}\right), \tag{38}$$

provided the boundary conditions (32)–(35) are "complementary" ([So, p. 121]).

To describe the complementary condition at the meeting point, i.e. at $x=0$, let $B(0,t,\partial_x,\partial_t)$ be the matrix of boundary conditions at $x=0$, i.e.

$$B(0,t,\partial_x,\partial_t)\,\vec{u}(0,t)=(0,0,0,0,\Phi,\Psi)\text{ where }\vec{u}=(u_j)\quad j=1,\cdots,6.$$

Then

$$B(0,t,i\tau,p)=\begin{bmatrix}1&0&-1&0&0&0\\0&1&0&-1&0&0\\0&0&1&0&-1&0\\0&0&0&1&0&-1\\i\tau b_{51}&i\tau b_{52}&i\tau b_{53}&i\tau b_{54}&0&0\\0&0&i\tau b_{63}&i\tau b_{64}&i\tau b_{65}&i\tau b_{66}\end{bmatrix}$$

where b_{ij}, $i=5,6$, $1\le j\le 6$, are the constants defined in (33)-(34). Let $\mathcal{L}(x,t,\partial_x,\partial_t)$ be the matrix of the system (29) and let $L\equiv\det\mathcal{L}(x,t,i\tau,p)=\prod_j(p+D_j\tau^2)$, and $\hat{\mathcal{L}}(x,t,i\tau,p)\equiv L\mathcal{L}^{-1}(x,t,i\tau,p)$. Then $\hat{\mathcal{L}}$ is a diagonal matrix and the element in row k is $\prod_{j\neq k}(p+D_j\tau^2)$. We note that the parabolicity condition is fulfilled since $D_j\tau^2\ge\frac{1}{\max_{1\le l\le 3}\sup_x|p^0_{lx}|^2}\tau^2$ (see

[So, p. 8]). The complementary condition at $x = 0$ is fulfilled if the rows of the matrix $A(0,t,i\tau,p) = B(0,t,i\tau,p)\hat{\mathcal{L}}(0,t,i\tau,p)$ are independent modulo the polynomial

$$\prod_j (\tau - i\sqrt{p/D_j}) \text{ when } \Re(p) \geq 0 \text{ and } |p|^2 > 0 \text{ (see [So, p. 11]).}$$

We must therefore verify that the homogeneous system of 6 equations with 6 unknowns given by

$$(a,b,c,d,e,f) \cdot A(0,t,i\tau,p) = \vec{0}$$

has the unique solution $(a,b,c,d,e,f) = \vec{0}$, mod $\prod_j(\tau - i\sqrt{p/D_j})$. Explicitly, we must verify that, mod $\prod_j(\tau - i\sqrt{p/D_j})$,

$$(a + e\, i\tau b_{51}) \prod_{j\neq 1} (\tau - i\sqrt{p/D_j}) = 0,$$

$$(b + e\, i\tau b_{52}) \prod_{j\neq 2} (\tau - i\sqrt{p/D_j}) = 0,$$

$$(-a + c + e\, i\tau b_{53} + f\, i\tau b_{64}) \prod_{j\neq 3} (\tau - i\sqrt{p/D_j}) = 0,$$

$$(-b + d + e\, i\tau b_{54} + f\, i\tau b_{64}) \prod_{j\neq 4} (\tau - i\sqrt{p/D_j}) = 0,$$

$$(-c + f\, i\tau b_{65}) \prod_{j\neq 5} (\tau - i\sqrt{p/D_j}) = 0,$$

$$(-d + f\, i\tau b_{66}) \prod_{j\neq 6} (\tau - i\sqrt{p/D_j}) = 0,$$

has $(a,b,c,d,e,f) = \vec{0}$ as its unique solution or, equivalently, that $(a,b,c,d,e,f) = \vec{0}$ is the only vector satisfying

$$(a - e\sqrt{p/D_1} b_{51}) = 0,$$

$$(b - e\sqrt{p/D_2}b_{52}) = 0,$$

$$(-a + c - \sqrt{p/D_3}(e\,b_{53} + f\,b_{63})) = 0,$$

$$(-b + d - \sqrt{p/D_4}(e\,b_{54} + f\,b_{64})) = 0,$$

$$(-c - f\sqrt{p/D_5}b_{65}) = 0,$$

$$(-d - f\sqrt{p/D_6}b_{66}) = 0.$$

Thus, it suffices to show that

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 & -\sqrt{p/D_1}b_{51} & 0 \\ 0 & 1 & 0 & 0 & -\sqrt{p/D_2}b_{52} & 0 \\ -1 & 0 & 1 & 0 & -\sqrt{p/D_3}b_{53} & -\sqrt{p/D_3}b_{63} \\ 0 & -1 & 0 & 1 & -\sqrt{p/D_4}b_{54} & -\sqrt{p/D_4}b_{64} \\ 0 & 0 & -1 & 0 & 0 & -\sqrt{p/D_5}b_{65} \\ 0 & 0 & 0 & -1 & 0 & -\sqrt{p/D_6}b_{66} \end{bmatrix} \neq 0. \tag{39}$$

¿From (39) and using that $D_1 = D_2 = \frac{1}{|p^0_{1x}|^2}$, $D_3 = D_4 = \frac{1}{|p^0_{2x}|^2}$ and $D_5 = D_6 = \frac{1}{|p^0_{3x}|^2}$, we find that the value of the determinant is

$$-p|p^0_{1x}|\,|p^0_{3x}|\,|p^0_{2x}|^2\frac{9\sqrt{3}}{8} \neq 0.$$

In the general case of arbitrary angles θ_j, it is easy to check that the condition for the system of boundary conditions at $x = 0$ to be complementary for the parabolic equations (29) is given by

$$(1 - \cos(\theta_1))(1 - \cos(\theta_2))(\sin(\theta_1 + \theta_2) + \sin(\theta_1) + \sin(\theta_2)) \neq 0\,,$$

which is satisfied if $0 < \theta_j < \pi$, $j = 1, 2$.

The complementary condition at $x = 1$ is easier to verify as the system decouples into three 2×2 subsystems. When $\alpha_j = 90^\circ$, the matrix $B(1, t, i\tau, p)$ is given by

$$B(1,t,i\tau,p) = \begin{bmatrix} \partial_{x_1} b(p_1^0) & \partial_{x_2} b(p_1^0) & 0 & 0 & 0 & 0 \\ -i\tau\partial_{x_2} b(p_1^0) & i\tau\partial_{x_1} b(p_1^0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_{x_1} b(p_2^0) & \partial_{x_2} b(p_2^0) & 0 & 0 \\ 0 & 0 & -i\tau\partial_{x_2} b(p_2^0) & i\tau\partial_{x_1} b(p_2^0) & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{x_1} b(p_3^0) & \partial_{x_2} b(p_3^0) \\ 0 & 0 & 0 & 0 & -i\tau\partial_{x_2} b(p_3^0) & i\tau\partial_{x_1} b(p_3^0) \end{bmatrix}$$

so that the rows of $B(1, t, i\tau, p)\hat{\mathcal{L}}(1, t, i\tau, p)$ are independent modulo $\prod_j(\tau - i\sqrt{p/D_j})$ if the determinant of

$$\begin{bmatrix} \partial_{x_1} b(p_1^0) & \partial_{x_2} b(p_1^0) & 0 & 0 & 0 & 0 \\ \sqrt{\frac{p}{D_1}}\partial_{x_2} b(p_1^0) & -\sqrt{\frac{p}{D_2}}\partial_{x_1} b(p_1^0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_{x_1} b(p_2^0) & \partial_{x_2} b(p_2^0) & 0 & 0 \\ 0 & 0 & \sqrt{\frac{p}{D_3}}\partial_{x_2} b(p_2^0) & -\sqrt{\frac{p}{D_4}}\partial_{x_1} b(p_2^0) & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{x_1} b(p_3^0) & \partial_{x_2} b(p_3^0) \\ 0 & 0 & 0 & 0 & \sqrt{\frac{p}{D_5}}\partial_{x_2} b(p_3^0) & -\sqrt{\frac{p}{D_6}}\partial_{x_1} b(p_3^0) \end{bmatrix}$$

is non-zero.

But using that $D_1 = D_2$, we find

$$\det \begin{bmatrix} \partial_{x_1} b(p_1^0) & \partial_{x_2} b(p_1^0) \\ \sqrt{\frac{p}{D_1}}\partial_{x_2} b(p_1^0) & -\sqrt{\frac{p}{D_2}}\partial_{x_1} b(p_1^0) \end{bmatrix} = -\sqrt{\frac{p}{D_1}}\left((\partial_{x_1} b(p_1^0))^2 + (\partial_{x_2} b(p_1^0))^2\right)$$

$$\neq 0,$$

and similarly for the determinant of the other 2×2 submatrices. Thus, the Neumann boundary condition (35) is complementary. In a similar way, one can show that the boundary conditions at $x = 1$ in (28) (with arbitrary angles α_j) are complementary as long as $(\partial_{x_1} b(p_j^0), \partial_{x_2} b(p_j^0)) \cdot p_{jx}^0 \neq 0$, $j = 1, 2, 3$. This means that the complementary condition is satisfied at $x = 1$ if $\alpha_j \neq 0$, i.e. as long as the curves are not tangent to $\partial\Omega$. Thus given $\bar{u}_j \in X_j$, $j = 1, .., 6$, there exists a unique solution to (29)–(35) satisfying (38).

Next, using (38), we show that the operator $\mathcal{R}$ maps $\prod_{j=1}^{6} X_j$ into $\prod_{j=1}^{6} X_j$. For this we first find a bound on the C^α norm of f_1; similar bounds hold for $|f_j|_\alpha$, $2 \le j \le 6$. Using the convention that the $C^{2+\alpha}$ norm of a vector is the sum of the norms of its components, it follows that for $\bar{u}_j \in X_j$, we have $|\bar{p}_j|_{2+\alpha} \le 2M$ and hence

$$|\bar{p}_{jx}| \ge |\frac{p_{jx}^0}{2}| > \frac{\delta}{2} \qquad \text{for } T \le t_0 = t_0(M, \delta), \tag{40}$$

where $\delta = \min_j \inf_x |p_{jx}^0(x)|$. Therefore, using the identity

$$\frac{1}{f^2(t)} - \frac{1}{f^2(0)} = -(f(t) - f(0)) \left(\frac{1}{f^2(0) f(t)} + \frac{1}{f(0) f^2(t)} \right) \tag{41}$$

and that $||\bar{p}_{1x}| - |p_{1x}^0|| \le |\bar{p}_{1x} - p_{1x}^0| \le |\bar{p}_{1x} - p_{1x}^0|_\infty$, we have

$$\left| \frac{1}{|\bar{p}_{1x}|^2} - \frac{1}{|p_{1x}^0|^2} \right|_\alpha \le \frac{C}{\delta^3} |\bar{p}_{1x} - p_{1x}^0|_\alpha \ . \tag{42}$$

Using (31) and (42), it follows that for $T \le t_0$

$$\begin{aligned} |f_1|_\alpha &\le \left(\left| \frac{1}{|\bar{p}_{1x}|^2} - \frac{1}{|p_{1x}^0|^2} \right|_\alpha \right) |\bar{u}_1|_{2+\alpha} \\ &\le C_\delta |\bar{p}_1|_{2+\alpha} T^{\frac{1}{2}} |\bar{u}_1|_{2+\alpha} \\ &\le C_\delta M^2 T^{\frac{1}{2}} \end{aligned} \tag{43}$$

which can be made arbitrarily small.

Next, we find a bound for the $C^{1+\alpha}$ norm of the function Φ defined in (36) (a bound for Ψ is obtained in a similar way). For this, we shall use the following identity which holds for any two functions h_1 and h_2 defined on $[0, T]$:

$$h_1 h_2 - h_1^0 h_2 - h_1 h_2^0 = (h_1 - h_1^0)(h_2 - h_2^0) - h_1^0 h_2^0 \tag{44}$$

with our convention that $h_1^0 = h_1(0)$ and $h_2^0 = h_2(0)$. Then

$$
\begin{aligned}
|\Phi|_{1+\alpha} &= |-\bar{p}_{1x}\cdot\bar{p}_{2x} - \frac{1}{2}|\bar{p}_{1x}||\bar{p}_{2x}| + \bar{B}_5|_{1+\alpha} \\
&= \left| -(\bar{p}_{1x} - p_{1x}^0)(\bar{p}_{2x} - p_{2x}^0) + p_{1x}^0 p_{2x}^0 - \frac{1}{2}(|\bar{p}_{1x}| - |p_{1x}^0|)(|\bar{p}_{2x}| - |p_{2x}^0|) + \frac{1}{2}|p_{1x}^0||p_{2x}^0| \right|_{1+\alpha} \\
&\le |\bar{p}_1|_{2+\alpha}|\bar{p}_2|_{2+\alpha}(T^{1+\alpha} + 2T^{\frac{1+\alpha}{2}}) + |(|\bar{p}_{1x}| - |p_{1x}^0|)(|\bar{p}_{2x}| - |p_{2x}^0|)|_{1+\alpha} \\
&\le C|\bar{p}_1|_{2+\alpha}|\bar{p}_2|_{2+\alpha}T^{\frac{1+\alpha}{2}} + C|\bar{p}_1|_{2+\alpha}|\bar{p}_2|_{2+\alpha}(T^{1+\alpha} + T^{\frac{1+\alpha}{2}}) \\
&\le CM^2T^{\frac{1+\alpha}{2}},
\end{aligned}
$$

where in the first inequality we have used the compatibility conditions for the initial data. On the other hand, at $x = 1$ we have (c.f. (37))

$$
\begin{aligned}
|\Xi_1|_{1+\alpha} &= \left| -\bar{p}_{1x}\cdot\vec{T}(\bar{p}_1) + \bar{p}_{1x}\cdot\vec{T}(p_1^0) \right|_{1+\alpha} \\
&= \left| \bar{p}_{1x}\int_0^1 D\vec{T}(\lambda p_1^0 + (1-\lambda)\bar{p}_1)(\bar{p}_1 - p_1^0)d\lambda \right|_{1+\alpha} \\
&\le C(|b|_{2+\alpha})|p_1|_{2+\alpha}^2(T + T^{\frac{1-\alpha}{2}}) \\
&\le C(|b|_{2+\alpha})M^2T^{\frac{1-\alpha}{2}}.
\end{aligned}
$$

with similar bounds for $|\Xi_2|_{1+\alpha}$ and $|\Xi_3|_{1+\alpha}$. Finally, since $b(p_1^0) = 0$

$$
\begin{aligned}
|\Xi_4|_{1+\alpha} &= |-b(\bar{p}_1) + Db(p_1^0)\bar{p}_1|_{1+\alpha} \\
&\le |Db(p_1^0)p_1^0|_{1+\alpha} + |Db(p_1^0)(\bar{p}_1 - p_1^0)|_{1+\alpha} + |\int_0^1 Db(\lambda p_1^0 + (1-\lambda)\bar{p}_1)(\bar{p}_1 - p_1^0)d\lambda|_{1+\alpha} \\
&\le |Db(p_1^0)p_1^0|_{1+\alpha} + C(|b|_{2+\alpha})M^2T^{\frac{1-\alpha}{2}}
\end{aligned}
$$

and similarly for $|\Xi_5|_{1+\alpha}$ and $|\Xi_6|_{1+\alpha}$. Putting this together in (38), we find for $T \le t_0$

$$
\sum_{j=1}^{6}|u_j|_{2+\alpha} \le C_s\left(\sum_{j=1}^{6}|u_j^0|_{2+\alpha} + \sum_{j=1}^{3}|Db(p_j^0)p_j^0|_{1+\alpha} + C(|b|_{2+\alpha},\delta)M^2T^{\frac{1-\alpha}{2}}\right).
$$

Hence, choosing

$$M \equiv 2C_3\left(\sum_{j=1}^{6}|u_1^0|_{2+\alpha} + \sum_{j=1}^{3}|Db(p_j^0)p_j^0|_{1+\alpha}\right),$$

we conclude that there exists a time $t_1 = t_1(M, |b|_{2+\alpha}, \delta) \leq t_0$ small enough so that

$$\sum |u_j|_{2+\alpha} \leq M \quad \text{for } T \leq t_1, \tag{45}$$

i.e. $\mathcal{R}$ maps $\prod X_j$ into itself.

Next we prove that the map $\mathcal{R}$ is a contraction. Let $(\bar{u}_j)$ and $(\bar{v}_j) \in \prod_j X_j$ with $T \leq t_1$, let $(u_j) = \mathcal{R}(\bar{u}_j)$ and $v_j = \mathcal{R}(\bar{v}_j)$ be the associated solutions to the linearized problem (29)–(35) and let $\omega_j = u_j - v_j$. Moreover, let p_j, q_j, $j = 1, 2, 3$ be the associated position vectors, ($q_1 = (v_1, v_2)$, etc.) and let $Z_j = p_j - q_j$. Then the ω_j's solve

$$\omega_{jt} - D_j\omega_{jxx} = g_j \tag{46}$$

$$\omega_j(x, 0) \equiv 0 \qquad (p_{jx}^0 = q_{jx}^0) \tag{47}$$

where D_j is as in (29) and

$$g_j = \begin{cases} \left(\frac{1}{|\bar{p}_{1x}|^2} - \frac{1}{|p_{1x}^0|^2}\right)\bar{u}_{jxx} - \left(\frac{1}{|\bar{q}_{1x}|^2} - \frac{1}{|q_{1x}^0|^2}\right)\bar{v}_{jxx} & j = 1, 2 \\ \left(\frac{1}{|\bar{p}_{2x}|^2} - \frac{1}{|p_{2x}^0|^2}\right)\bar{u}_{jxx} - \left(\frac{1}{|\bar{q}_{2x}|^2} - \frac{1}{|q_{2x}^0|^2}\right)\bar{v}_{jxx} & j = 3, 4 \\ \left(\frac{1}{|\bar{p}_{3x}|^2} - \frac{1}{|p_{3x}^0|^2}\right)\bar{u}_{jxx} - \left(\frac{1}{|\bar{q}_{3x}|^2} - \frac{1}{|q_{3x}^0|^2}\right)\bar{v}_{jxx} & j = 5, 6, \end{cases}$$

with the corresponding boundary conditions

$$\begin{aligned} B_j(0, t, \omega) &= \omega_j(0, t) - \omega_{j+2}(0, t) = 0 \qquad 1 \leq j \leq 4 \\ B_5(0, t, \omega) &\equiv b_{51}\omega_{1x} + b_{52}\omega_{2x} + b_{53}\omega_{3x} + b_{54}\omega_{4x} = \Phi(\bar{p}_{1x}, \bar{p}_{2x}) - \Phi(\bar{q}_{1x}, \bar{q}_{2x}) \\ B_6(0, t, \omega) &\equiv b_{63}\omega_{3x} + b_{64}\omega_{4x} + b_{65}\omega_{5x} + b_{66}\omega_{6x} = \Psi(\bar{p}_{2x}, \bar{p}_{3x}) - \Psi(\bar{q}_{2x}, \bar{q}_{3x}) \end{aligned}$$

and

$$\begin{aligned} B_j(1, t, \omega) &\equiv Z_{jx} \cdot \vec{T}(p_1^0) = \Xi_j(\bar{p}_j) - \Xi_j(\bar{q}_j) \qquad j = 1, 2, 3 \\ B_{j+3}(1, t, \omega) &\equiv Db(p_j^0) \cdot Z_j = \Xi_{j+3}(\bar{p}_j) - \Xi_{j+3}(\bar{q}_j) \qquad j = 1, 2, 3. \end{aligned} \tag{48}$$

Since this is a linear parabolic system that satisfies the complementary conditions and the compatibility conditions, we shall again use Schauder-type estimates to show that

$$\sum_{j=1}^{6} |\omega_j|_{2+\alpha} \leq \frac{1}{2} \sum_{j=1}^{6} |\bar{\omega}_j|_{2+\alpha} \qquad \text{for } T \leq t_2,$$

where $t_2 \leq t_1$ and t_1 is as in (45).

First using (41) and (47), we find a bound for g_1 in C^α:

$$\begin{aligned} |g_1|_\alpha &\leq \left| \left(\frac{1}{|\bar{p}_{1x}|^2} - \frac{1}{|p^0_{1x}|^2} \right) \bar{\omega}_{1xx} \right|_\alpha + \left| \left(\frac{1}{|\bar{p}_{1x}|^2} - \frac{1}{|\bar{q}_{1x}|^2} \right) \bar{v}_{1xx} \right|_\alpha \\ &\leq C_\delta \, (T^{\frac{1}{2}} |\bar{p}_1|_{2+\alpha} |\bar{\omega}_1|_{2+\alpha} + |\bar{v}_1|_{2+\alpha} |\bar{Z}_{1x}|_\alpha) \\ &\leq C_\delta \, M \, T^{\frac{1}{2}} |\bar{Z}_1|_{2+\alpha}. \end{aligned}$$

A similar bound holds for g_j, $2 \leq j \leq 6$. Next, we use (44) and the identity

$$\begin{aligned} (h_1 - h^0_1)(h_2 - h^0_2) - (j_1 - j^0_1)(j_2 - j^0_2) &= (h_1 - j_1)(h_2 - j_2) + (h_1 - j_1)(j_2 - j^0_2) \\ &\quad + (h_2 - j_2)(j_1 - j^0_1) \end{aligned}$$

to obtain, at $x = 0$,

$$\begin{aligned} |\Phi(\bar{p}_{1x}, \bar{p}_{2x}) - \Phi(\bar{q}_{1x}, \bar{q}_{2x})|_{1+\alpha} &= |(\bar{q}_{1x} - q^0_{1x})(\bar{q}_{2x} - q^0_{2x}) - (\bar{p}_{1x} - p^0_{1x})(\bar{p}_{2x} - p^0_{2x})|_{1+\alpha} \\ &\leq |\bar{Z}_{1x} \bar{Z}_{2x}|_{1+\alpha} + |\bar{Z}_{1x} (\bar{p}_{2x} - p^0_{2x})|_{1+\alpha} + |\bar{Z}_{2x} (\bar{p}_{1x} - p^0_{1x})|_{1+\alpha} \\ &\leq CM(|\bar{Z}_1|_{2+\alpha} + |\bar{Z}_2|_{2+\alpha}) T^{\frac{1+\alpha}{2}}. \end{aligned}$$

The same type of bound can be obtained for the term involving Ψ. On the other hand at $x = 1$,

$$\begin{aligned} |\Xi_1(\bar{p}_1) - \Xi_1(\bar{q}_1)|_{1+\alpha} &= |(\bar{q}_{1x} - \bar{p}_{1x}) \vec{T}(\bar{p}_1) + \left(\vec{T}(\bar{q}_1) - \vec{T}(\bar{p}_1) \right) \bar{q}_{1x} + \vec{T}(p^0_1)(\bar{p}_{1x} - \bar{q}_{1x})|_{1+\alpha} \\ &\leq |\bar{Z}_{1x} \int_0^1 D\vec{T}(\lambda \bar{p}_1 + (1-\lambda) p^0_1)(\bar{p}_1 - p^0_1) d\lambda|_{1+\alpha} \\ &\quad + |\int_0^1 D\vec{T}(\lambda \bar{q}_1 + (1-\lambda) \bar{p}_1)(\bar{q}_1 - \bar{p}_1) \bar{q}_{1x}|_{1+\alpha} \\ &\leq C(|b|_{2+\alpha}) |Z_1|_{2+\alpha} M T^{\frac{1-\alpha}{2}}. \end{aligned}$$

A similar bound holds for $|\Xi_2(\bar{p}_2) - \Xi_2(\bar{q}_2)|_{1+\alpha}$ and $|\Xi_3(\bar{p}_3) - \Xi_3(\bar{q}_3)|_{1+\alpha}$. As for Ξ_4, we have

$$\begin{aligned} |\Xi_4(\bar{p}_1) - \Xi_4(\bar{q}_1)|_{1+\alpha} &= |b(\bar{q}_1) - b(\bar{p}_1) + Db(p_1^0)(\bar{p}_1 - \bar{q}_1)|_{1+\alpha} \\ &= |\int_0^1 Db(\lambda\bar{q}_1 + (1-\lambda)\bar{p}_1)(\bar{q}_1 - \bar{p}_1)d\lambda + Db(p_1^0)\bar{Z}_1|_{1+\alpha} \\ &\leq C(|b|_{2+\alpha})|Z_1|_{2+\alpha}T^{\frac{1-\alpha}{2}}. \end{aligned}$$

Again similar bounds hold for the terms involving Ξ_5 and Ξ_6. Therefore, putting this together and using the Schauder estimates for (46), it follows

$$\sum_{j=1}^{6} |\omega_j|_{2+\alpha} \leq C_s C(|b|_{2+\alpha}, \delta) M T^{\frac{1-\alpha}{2}} \sum_{j=1}^{6} |\bar{\omega}_j|_{2+\alpha}.$$

Thus $\mathcal{R}$ is a contraction for

$$T \leq t_2(M, |b|_{2+\alpha}, \delta)$$

where t_2 is such that

$$C_s C(|b|_{2+\alpha}, \delta) M t_2^{\frac{1-\alpha}{2}} \leq 1/2. \qquad \square$$

Remark 2 A word is in order to justify the possible non-existence of a solution when either $\alpha_j = 0$ or $\theta_j = 0$. We show this in a model case: we assume that Γ_1 is a graph so that its position is given by (x_1, u) where

$$u_t = \frac{u_{x_1 x_1}}{1 + u_{x_1}^2}. \qquad (49)$$

Let $x_1 = s(t)$ denote the intersection of Γ_1 with the boundary of $\Omega = \{x_2 > 0\}$ and let $\alpha_1 = 0$. Then $u \geq 0$ and the boundary conditions for u are

$$u(s(t), t) = 0$$

and

$$u_{x_1}(s(t), t) = 0,$$

which contradict Hopf's Lemma. A similar argument works at the meeting point of the three curves: for the model problem in this case, one can take one of the curves to coincide with the x_1-axis and the other two to be graphs symmetric about this axis. In this model, the graphs stay symmetric, and we are in the same situation as above.

As can be seen from the proof of Theorem 1, the method used to obtain local existence is simple and quite general. Indeed it is enough to show that the problem is parabolic and that the linearized boundary conditions are complementary. Therefore it can be used in a wide variety of physical systems. As a further example we consider another problem in the theory of phase transitions, namely eutectic solidification ([W], [K]).

In lamellar eutectics two solid phases grow into a liquid phase. We model this process by considering two curves (the solid-liquid interfaces) meeting at a point with a prescribed angle, moving normally with a given speed whose dominant contribution is proportional to the curvature of each curve. As the two curves evolve, the locus of the meeting point traces out a third curve (the solid-solid interface) that should maintain a fixed angle with the solid-liquid interfaces at the meeting point (see Figure 4). The system of equations for the position vectors p_j of these two curves is therefore given by

$$p_{jt} = \frac{p_{jxx}}{|p_{jx}|^2} + a(x,t,p_j,p_{jx}) \quad x \in [0,1]\,,\ j=1,2 \tag{50}$$

$$p_j(x,0) = p_j^0(x)$$

$$p_1(0,t) = p_2(0,t)$$

$$\frac{p_{1x}}{|p_{1x}|} \cdot \frac{p_{1t}}{|p_{1t}|} = \cos(\theta_1) \quad \text{at } x=0$$

$$\frac{p_{1x}}{|p_{1x}|} \cdot \frac{p_{2x}}{|p_{2x}|} = \cos(\theta_1+\theta_2) \quad \text{at } x=0$$

$$\frac{p_{jx}}{|p_{jx}|} \cdot \frac{\vec{T}(p_j)}{|\vec{T}(p_j)|} = \cos(\alpha_j) \quad \text{at } x=1$$

$$b(p_j) = 0 \quad \text{at } x=1\,,$$

where $\vec{T}$ and b are as before, $0 < \theta_j < \pi$ and a is any smooth function. In a very similar way as above, we can prove short-time existence and uniqueness for this problem as long as $\alpha_j \neq 0$ and $\theta_1 + \theta_2 \neq \pi$. Since the proof is similar to that of Theorem 1, we will only give a sketch of the proof. Also, for the sake of simplicity, we will only consider the special case that $a \equiv 0$ since it is clear that the argument works equally well for any smooth function a.

Sketch of proof: The main difference with the system (26) is that the boundary conditions at $x = 0$ contain time derivatives. To overcome this difficulty we shall replace p_{jt} in the boundary condition at $x = 0$ with the help of (50) and study the problem satisfied by $q_j = p_{jx}$ around $x = 0$. The system of equations for q_j is:

$$q_{jt} = \frac{q_{jxx}}{|q_j|^2} - 2q_{jx}\frac{q_j \cdot q_{jx}}{|q_j|^4}. \tag{51}$$

To find the boundary conditions at the triple junction, we observe that $p_{1t}(0,t) = p_{2t}(0,t)$ so that using (50), we find:

$$\frac{q_{1x}}{|q_1|^2} = \frac{q_{2x}}{|q_2|^2} \qquad \text{at } x = 0. \tag{52}$$

Moreover

$$\frac{p_{1t}}{|p_{1t}|} = \frac{p_{1xx}}{|p_{1xx}|},$$

and hence the other boundary conditions at $x = 0$ become

$$\frac{q_1}{|q_1|} \cdot \frac{q_{1x}}{|q_{1x}|} = \cos(\theta_1) \quad \text{at } x = 0 \tag{53}$$

$$\frac{q_1}{|q_1|} \cdot \frac{q_2}{|q_2|} = \cos(\theta_1 + \theta_2) \quad \text{at } x = 0. \tag{54}$$

This is now a second order parabolic system with boundary conditions involving only q and q_x. If we linearize (51)-(54), we can show that the boundary conditions at the triple junction are complementary provided that the determinant of the matrix

$$\begin{bmatrix} -D_1\sqrt{pD_2} & 0 & -\sqrt{pD_2}(u_1^0 - \cos(\theta_1)\frac{1}{\sqrt{D_1}|q_{1x}^0|}u_{1x}^0) & \sqrt{D_1 D_2}u_3^0 - \cos(\theta_1+\theta_2)D_1 u_1^0 \\ 0 & -D_1\sqrt{pD_2} & -\sqrt{pD_2}(u_2^0 - \cos(\theta_1)\frac{1}{\sqrt{D_1}|q_{1x}^0|}u_{2x}^0) & \sqrt{D_1 D_2}u_4^0 - \cos(\theta_1+\theta_2)D_1 u_2^0 \\ -\sqrt{pD_1D_2} & 0 & 0 & -\sqrt{D_1}u_1^0 + \cos(\theta_1+\theta_2)\sqrt{D_2}u_3^0 \\ 0 & -\sqrt{pD_1D_2} & 0 & -\sqrt{D_1}u_2^0 + \cos(\theta_1+\theta_2)\sqrt{D_2}u_4^0 \end{bmatrix}$$

is nonzero. Here $D_1 = \frac{1}{|q_1^0|^2}$, $D_2 = \frac{1}{|q_2^0|^2}$, $q_1 = (u_1, u_2)$ and $q_2 = (u_3, u_4)$.

The value of the determinant is

$$\sqrt{p^3}\frac{1}{|q_1^0|^2}\frac{1}{|q_2^0|^3}\sin(\theta_1)(\cos(\theta_1+\theta_2)-1)(\cos(\theta_1)+\cos(\theta_2))$$

and hence the boundary conditions at the triple junction are complementary as long as $\theta_1+\theta_2\neq\pi$.

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