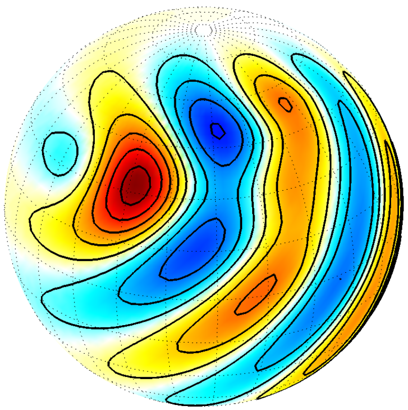
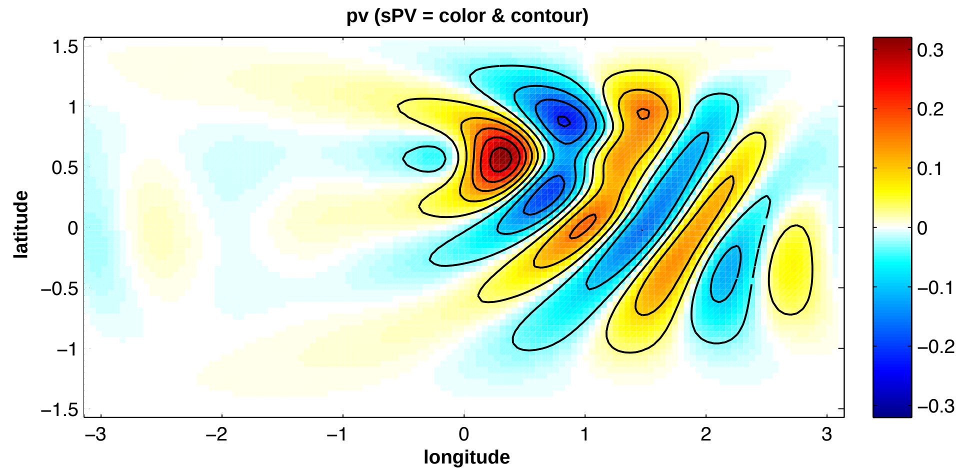
A PV Dynamics for Rotating Shallow Water on the Sphere

- ▷ search for a theory of balanced flow on the full sphere
- ▷ potential vorticity inversion & advection
- ▷ ray theory for short-scale waves





- ▷ David J Muraki, Kevin Mitchell & Andrea Blazenko; Simon Fraser University
- ▷ Chris Snyder; NCAR

NSERC CRSNG

Midlatitude Balanced Dynamics

Rotating Shallow Water (rSW) $\longrightarrow$ β-plane scaling

- winds, $\vec{u}$ & height field, $H = 1 + \epsilon\,\eta$
- longitude-latitude <u>local</u> coordinates, $(\lambda, \phi) \approx (\lambda_0, \phi_0)$

$$\epsilon^2 \frac{D\vec{u}}{Dt} \quad + \quad (1 + \beta(\phi - \phi_0))(\hat{r}_0 \times \vec{u}) \quad \sim \quad -\,\epsilon\,\nabla\eta$$

$$\epsilon\ \frac{D\eta}{Dt} \quad + \quad (1 + \epsilon\,\eta)\ \ (\nabla \cdot \vec{u}) \quad = \quad 0$$

- dimensionless parameters: Rossby number & Coriolis-β

$$\epsilon = \frac{\sqrt{gH_0}}{(2\Omega \sin\phi_0)\, r} \ll 1 \qquad ; \qquad \beta = \cot\phi_0$$

Midlatitude Quasigeostrophy (QG)

- *balanced* dynamics: NO fast waves, PV dynamics, $\epsilon \longrightarrow 0$ asymptotic limit
- restrict to short deformation scales: $\lambda - \lambda_0 = \epsilon x$; $\phi - \phi_0 = \epsilon y$
  - geostrophy: $\hat{r}_0 \times \vec{u} \sim -\,\epsilon\,\nabla\eta \quad \Rightarrow$ non-divergent winds
  - limit as $\epsilon \longrightarrow 0$, geostrophic degeneracy
  - advection & inversion of potential vorticity (PV)
- geometric obstacle: <u>local Rossby number singular at Equator, $\epsilon \longrightarrow \infty$</u>

rSW on the Full Sphere

Spherical Coordinates

- longitude-latitude global coordinates, (λ, ϕ)

$$\epsilon^2 \left\{ \frac{Du}{Dt} + v\hat{\lambda} \cdot \frac{D\hat{\phi}}{Dt} \right\} \quad - \; v \,\sin\phi \quad = \quad -\,\epsilon \,\frac{1}{\cos\phi}\,\eta_\lambda$$

$$\epsilon^2 \left\{ \frac{Dv}{Dt} + u\hat{\phi} \cdot \frac{D\hat{\lambda}}{Dt} \right\} \quad + \; u \,\sin\phi \quad = \quad -\,\epsilon \quad \eta_\phi$$

$$\epsilon \quad \frac{D\eta}{Dt} \; + \; \{1 + \epsilon\,\eta\} \,\frac{u_\lambda + (v\cos\phi)_\phi}{\cos\phi} \quad = \quad 0$$

- large Lamb parameter asymptotics

$$\epsilon = \left(\frac{4\Omega^2 r^2}{g H_0} \right)^{-1/2} \ll 1$$

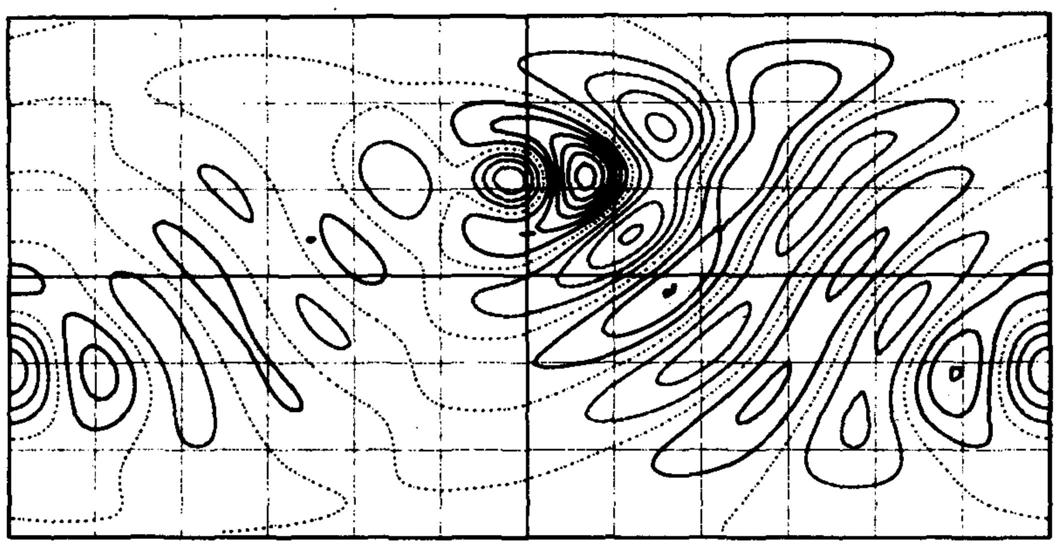
Balanced Dynamics

- *PV Inversion on a Hemisphere*, McIntyre/Norton 1999
  - landmark for PV as prognostic variable
  - non-divergent winds (at leading-order)
  - dynamical oddity: mirror symmetry across Equator
- on midlatitude scales, naïve geostrophic limit with $\epsilon \longrightarrow 0$ is now inconsistent!

A Case for Global Balanced Dynamics

Grose & Hoskins (1979)

- rSW flow past mountain at 30°N
  - steady, super-rotation zonal wind
- steady waves by Rayleigh-damped perturbations
  - perturbation vorticity after 17 days (with damping) & $\epsilon \approx 0.33$

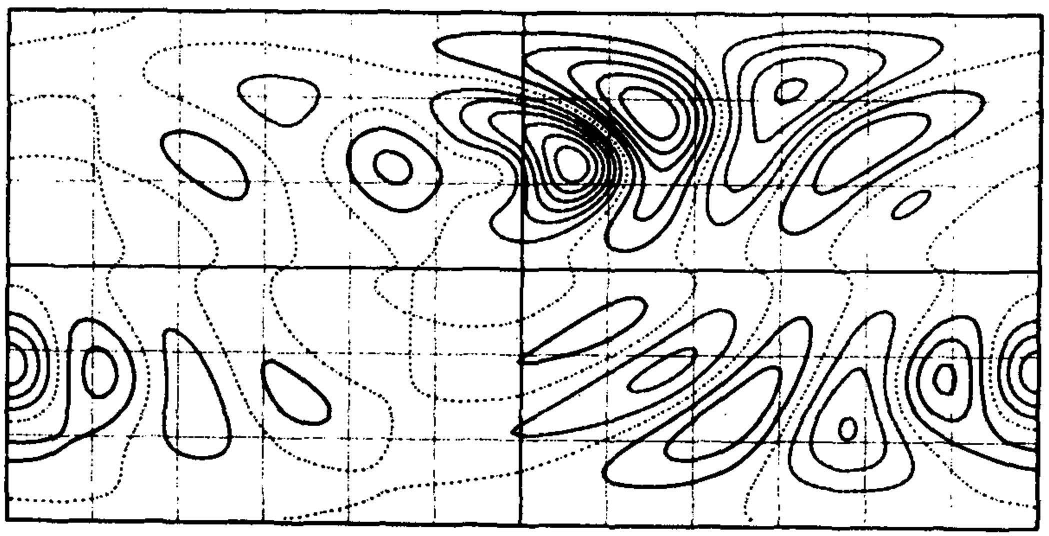


- waves propagate smoothly across Equator
- nearly non-divergent flow

A Case for Global Balanced Dynamics

Grose & Hoskins (1979)

- rSW flow past mountain at 30°N
  - steady, super-rotation zonal wind
- steady waves by Rayleigh-damped perturbations
  - perturbation height after 17 days (with damping) & $\epsilon \approx 0.33$



- essentially zero height disturbance at Equator
- non-divergent balance relations, with streamfunction ψ

$$u = -\,\epsilon\,\psi_{\phi} \qquad ; \qquad v = \epsilon\,\frac{1}{\cos\phi}\,\psi_{\lambda} \qquad ; \qquad \eta = \psi \sin\phi$$

rSW on the Sphere Redone

$$
\begin{aligned}
\epsilon^2 \frac{Du}{Dt} \quad & - v \sin\phi && \sim && - \epsilon \frac{1}{\cos\phi} \psi_\lambda \sin\phi && \\
\epsilon^2 \frac{Dv}{Dt} \quad & + u \sin\phi && \sim && - \epsilon\, \psi_\phi \sin\phi && - \epsilon\, \psi \cos\phi \\
\epsilon \frac{D\eta}{Dt} + \{1 + \epsilon\,\eta\} & \frac{u_\lambda + (v\cos\phi)_\phi}{\cos\phi} && = && 0 &&
\end{aligned}
$$

Geostrophic Degeneracy Restored

- on midlatitude scales, limit as $\epsilon \longrightarrow 0$ is consistently degenerate
- non-divergent balance relations
- β-effect displaced: meridional advection of planetary PV

Potential Vorticity

- total PV is advected quantity: $DQ/Dt = 0$

$$
Q = \sin\phi + \epsilon\, q = \frac{\sin\phi + \epsilon^2\, \hat{r} \cdot (\nabla \times \vec{u})}{1 + \epsilon\eta}
$$

PV Dynamics on the Sphere (sPV)

Inversion, Streamfunction & Advection

- PV-streamfunction relation with topography, $b(\lambda, \phi)$:

$$\epsilon^2 \nabla^2 \psi - (\sin^2 \phi)\, \psi = q - b \sin \phi$$

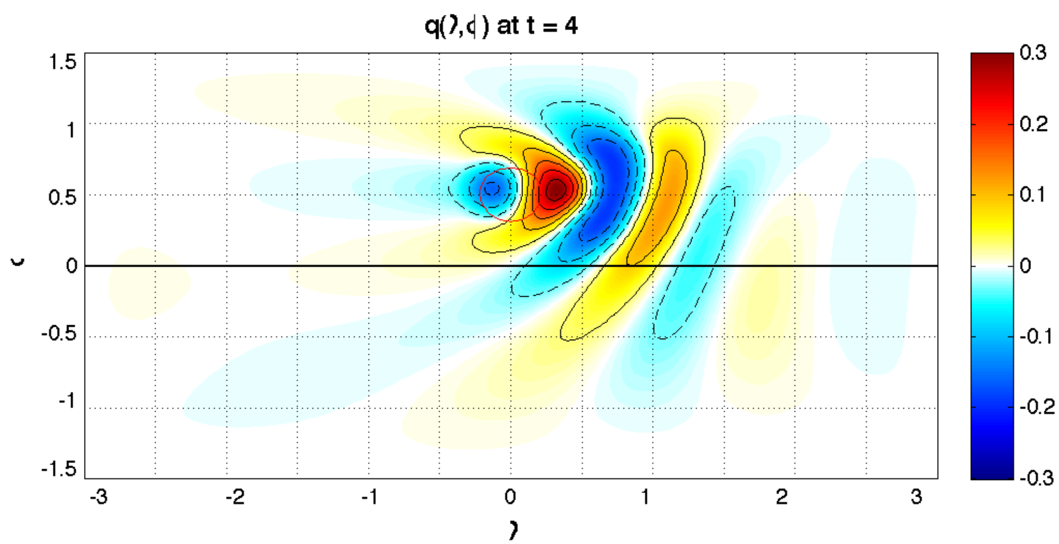
- balance relations for non-divergent winds:

$$u = -\,\epsilon\, \psi_\phi \qquad ; \qquad v = \epsilon \frac{1}{\cos \phi} \psi_\lambda \qquad ; \qquad \eta = \psi \sin \phi$$

- disturbance PV advection:

$$\epsilon \left\{ \frac{\partial q}{\partial t} + \frac{u}{\cos \phi} \frac{\partial q}{\partial \lambda} + v \frac{\partial q}{\partial \phi} \right\} + v \cos \phi = 0$$

Mountain Waves, Disturbance PV ($t = 4$)



PV Dynamics on the Sphere (sPV)

Inversion, Streamfunction & Advection

- PV-streamfunction relation with topography, $b(\lambda, \phi)$ & large-scale jet, $\bar{u}(\phi), \epsilon\bar{\eta}(\phi) = O(1)$:

$$\epsilon^2 \, (1 + \epsilon\bar{\eta}) \, \nabla^2 \, \psi - (\sin^2 \phi) \, \psi = (1 + \epsilon\bar{\eta})^2 \, q - b \, \sin \phi$$

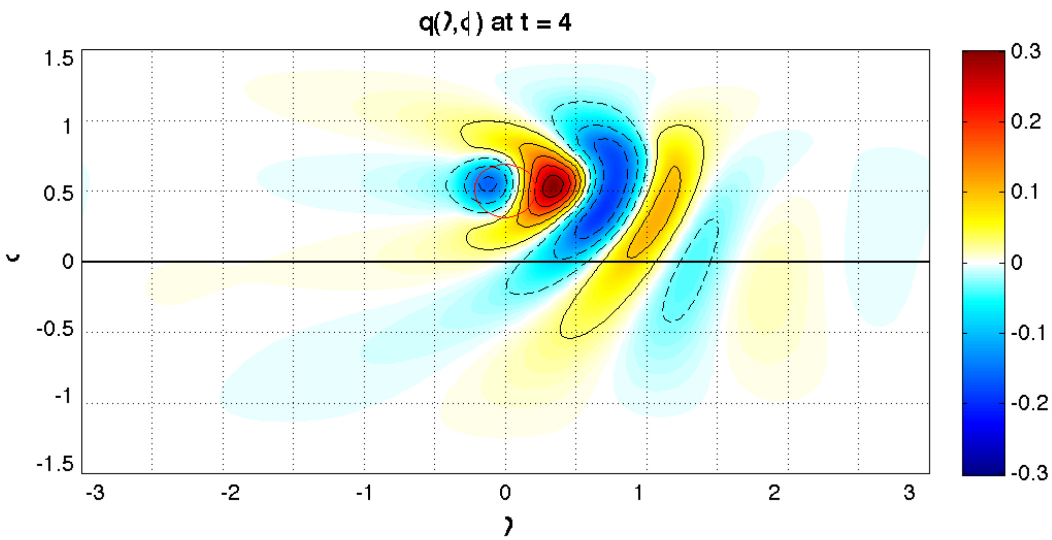
- balance relations for non-divergent winds:

$$u = -\, \epsilon \, \psi_\phi \qquad ; \qquad v = \epsilon \, \frac{1}{\cos\phi} \, \psi_\lambda \qquad ; \qquad \eta = \psi \, \sin\phi$$

- disturbance PV advection:

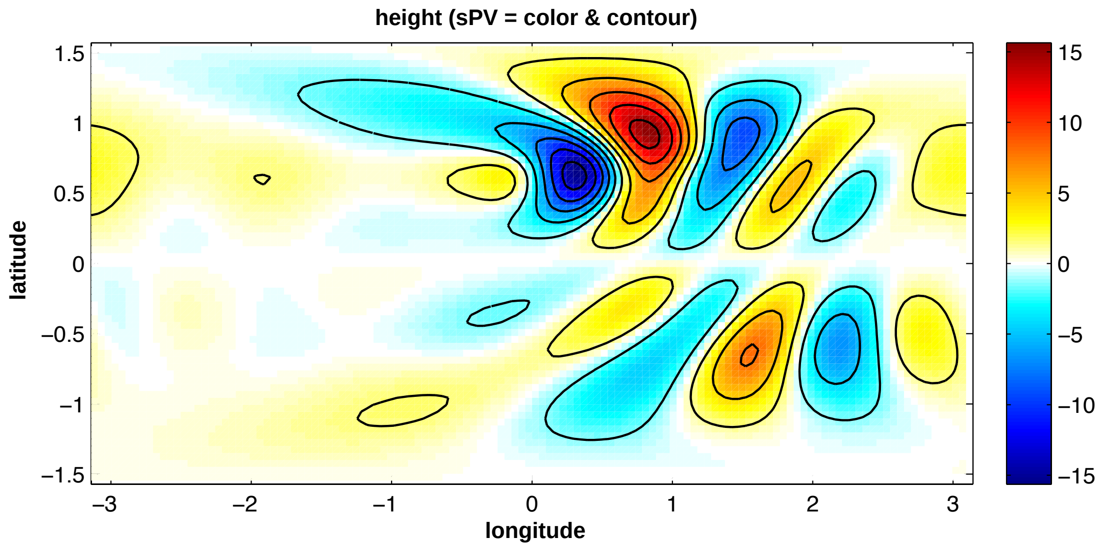
$$\epsilon \left\{ \frac{\partial q}{\partial t} + \frac{\bar{u} + u}{\cos\phi} \frac{\partial q}{\partial \lambda} + v \, \frac{\partial q}{\partial \phi} \right\} + v \left(\frac{\sin\phi}{1 + \epsilon\bar{\eta}} \right)_\phi = 0$$

Mountain Waves, Disturbance PV ($t = 4$)

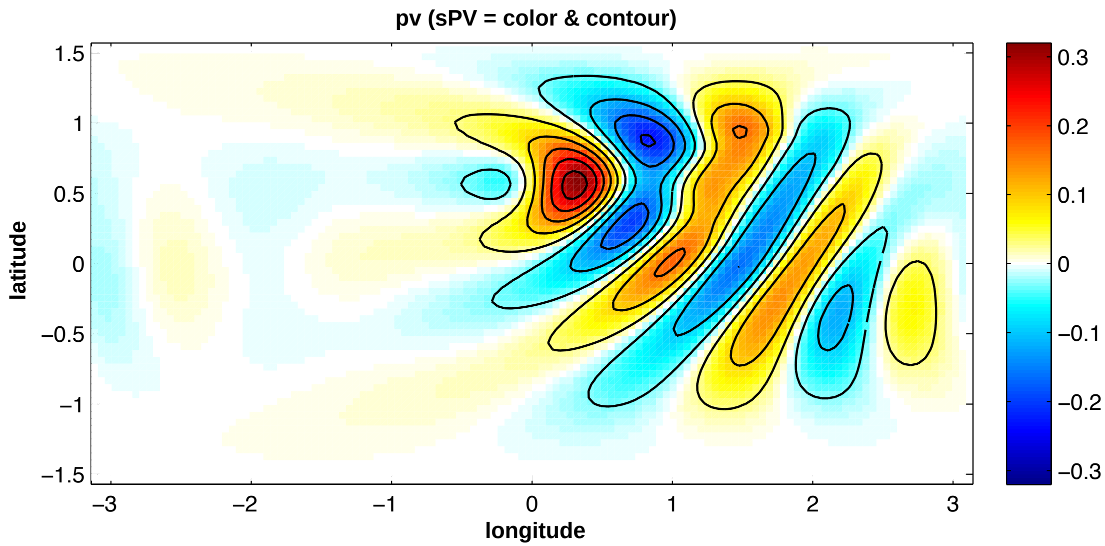


sPV on the Sphere: Computations

Mountain Waves, Disturbance Height ($t = 8$)

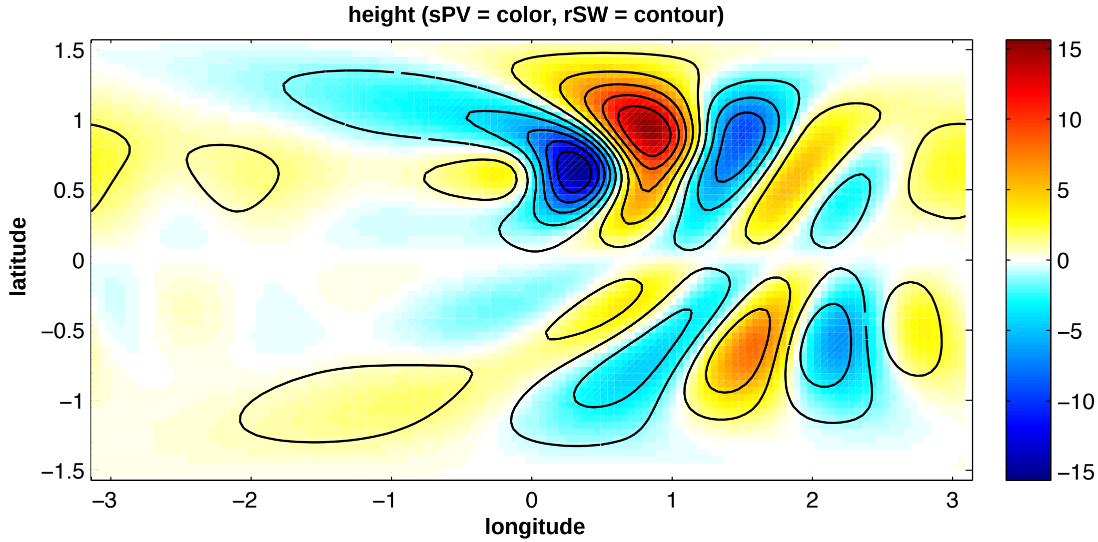


Mountain Waves, Disturbance PV ($t = 8$)

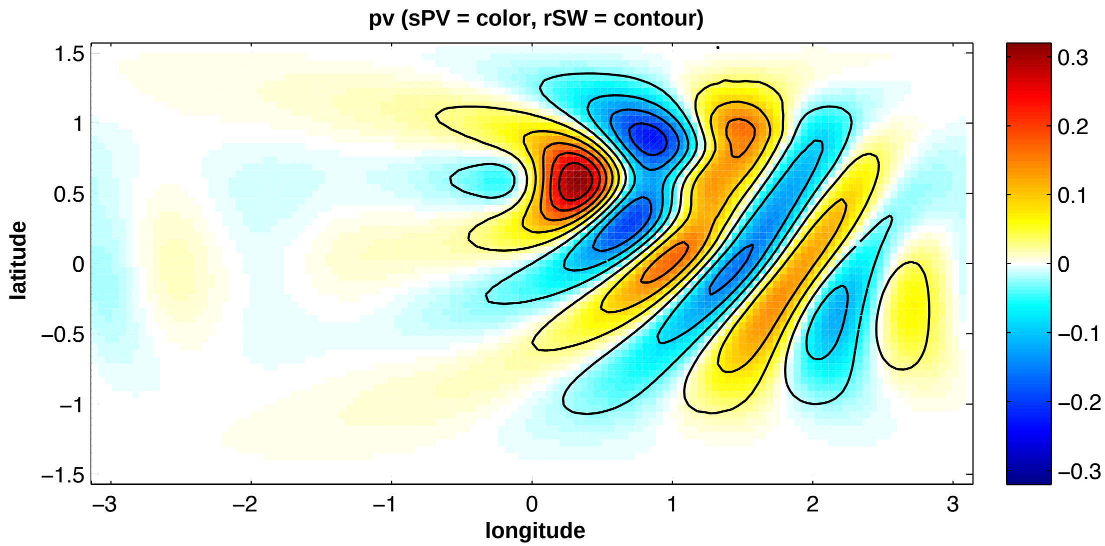


sPV on the Sphere: Comparison with rSW

Disturbance Height: sPV (colour) & rSW (contour)



Disturbance PV: $\epsilon \approx 0.337$



sPV is not a Global Asymptotic Theory?

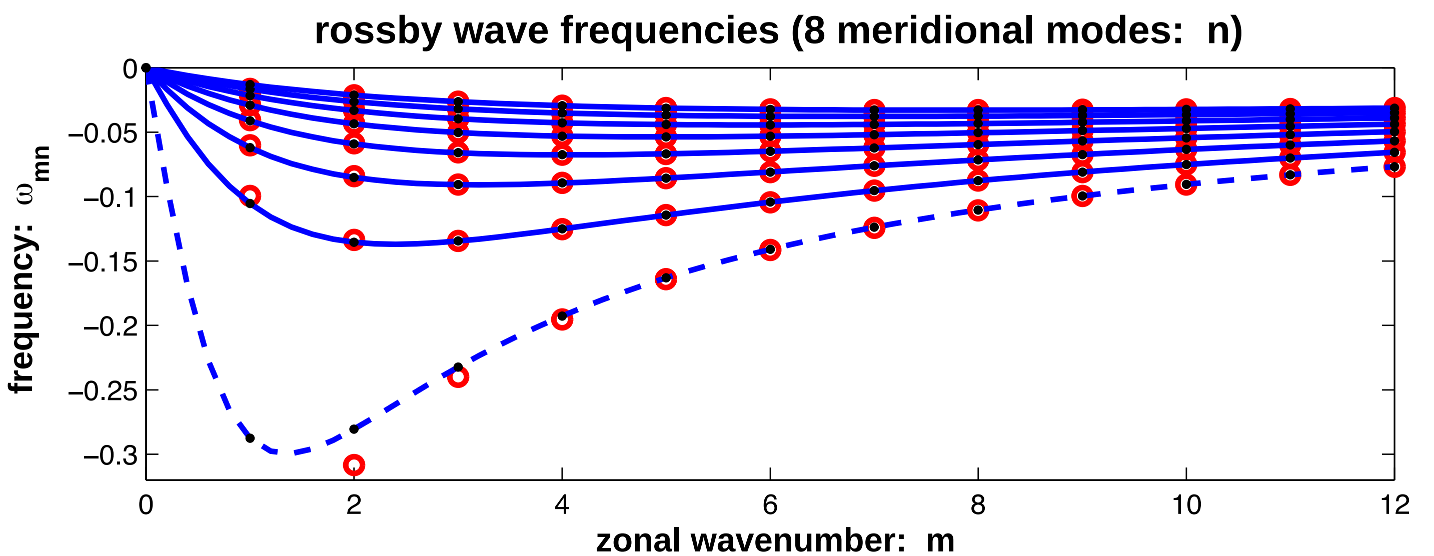
$$
\begin{aligned}
\epsilon^2 \frac{Du}{Dt} \qquad\qquad & - v \sin\phi \quad \sim \quad - \epsilon \frac{1}{\cos\phi} \psi_\lambda \sin\phi \\
\epsilon^2 \frac{Dv}{Dt} \qquad\qquad & + u \sin\phi \quad \sim \quad - \epsilon\, \psi_\phi \sin\phi \qquad - \epsilon\, \psi \cos\phi \\
\epsilon \frac{D\eta}{Dt} + \{1 + \epsilon\, \eta\} & \frac{u_\lambda + (v \cos\phi)_\phi}{\cos\phi} \quad = \quad 0
\end{aligned}
$$

Midlatitude Synoptic-Scale Truncation

- leading-order balance assumptions
  - on deformation scales $\Rightarrow$ $\partial/\partial\lambda, \partial/\partial\phi = O(1/\epsilon)$
  - at midlatitudes $\Rightarrow$ $\sin\phi \neq 0$
- there is NO expectation of asymptotic validity at the Equator
  - sPV is not *globally* accurate, but is well-posed
  - PV inversion & velocity-streamfunction relationship are non-singular

Q: So, Why are G&H Equatorial Crossings Faithfully Represented?

sPV Rossby Waves



Rotating Waves, $\psi(\lambda - ct, \phi)$

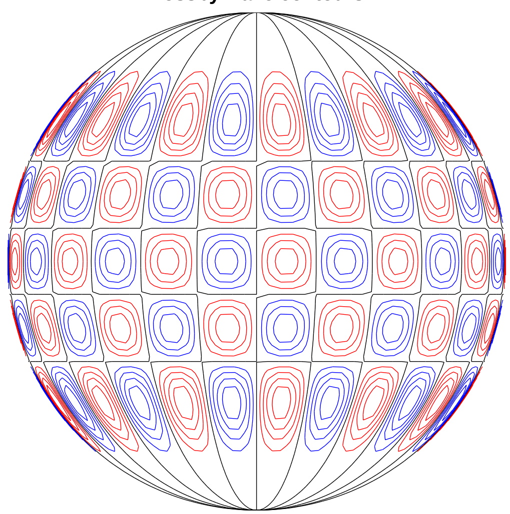
- ▷ ODE eigenfunction [Verkley (2007), Schubert (2008)] $\longrightarrow$ exact nonlinear sPV solutions ($\epsilon \approx 0.337$)

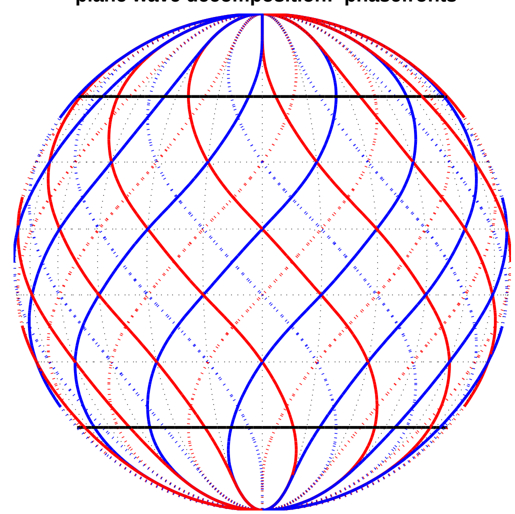
$$\epsilon^2 \nabla^2 \psi - (\sin^2 \phi)\, \psi + (1/c)\, \psi = 0 \quad ; \quad \psi(\pm\pi/2) = 0$$

- ▷ compare to modes for rSW on sphere (Margules, Hough, Longuet-Higgins, . . .)

- ▷ all sPV modes are slow, but longest planetary waves have $O(1)$ wavespeed errors

sPV Rossby Waves





Rotating Waves, $\psi(\lambda - ct, \phi)$

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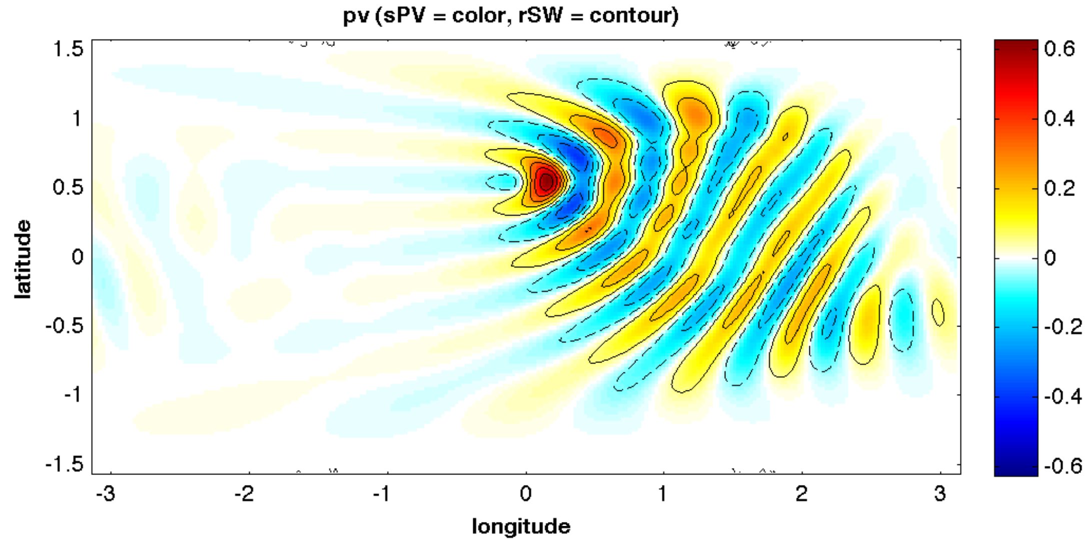
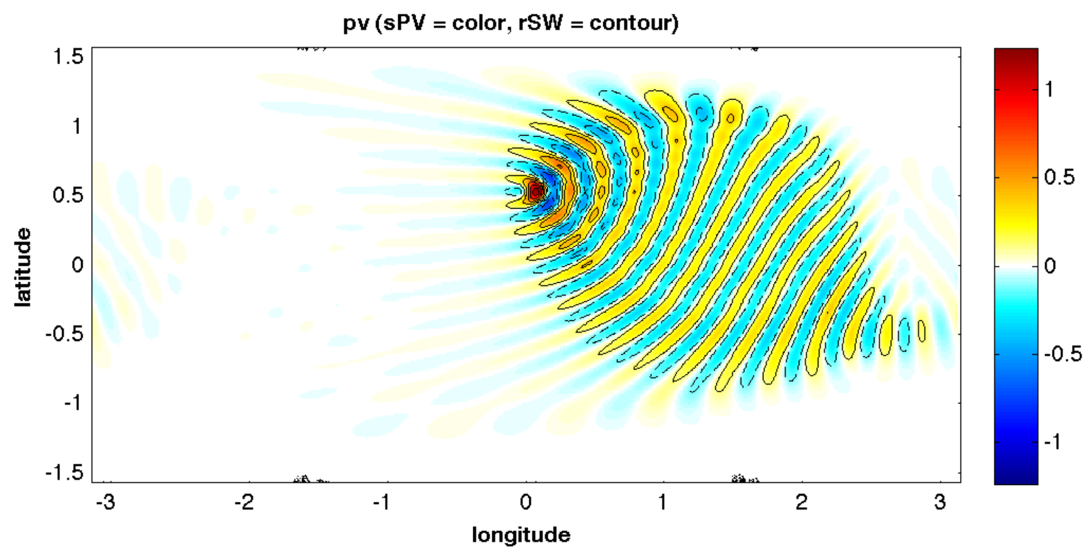
$$\epsilon^2 \nabla^2 \psi - (\sin^2 \phi)\, \psi + (1/c)\, \psi = 0 \quad ; \quad \psi(\pm\pi/2) = 0$$

- ▷ compare to modes for rSW on sphere (Margules, Hough, Longuet-Higgins, . . .)

- ▷ only longest planetary waves have $O(1)$ wavespeed errors

Wavepacket Limit of Short-Scale Waves: $\epsilon = \frac{1}{2}\,\epsilon_{\mathrm{gh}}$ & $\frac{1}{4}\,\epsilon_{\mathrm{gh}}$





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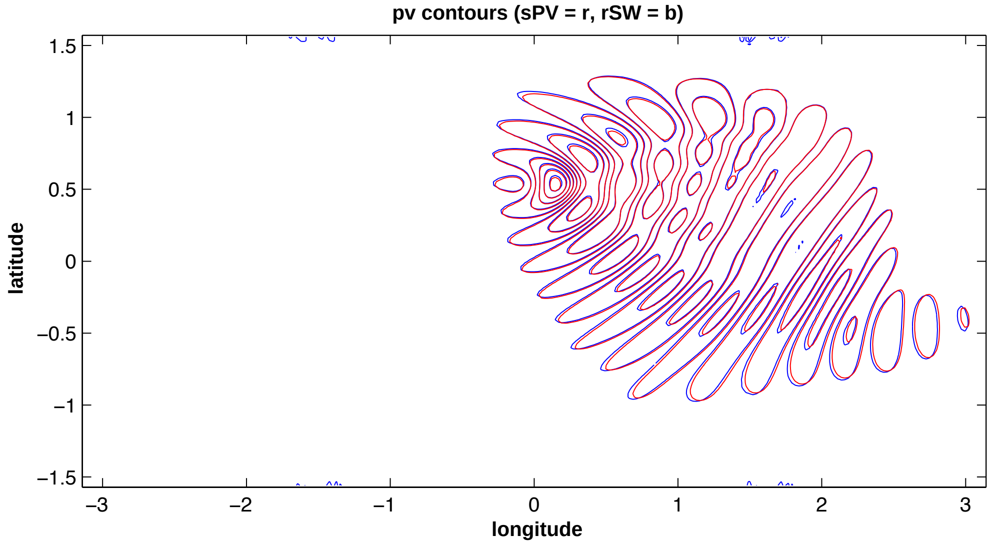
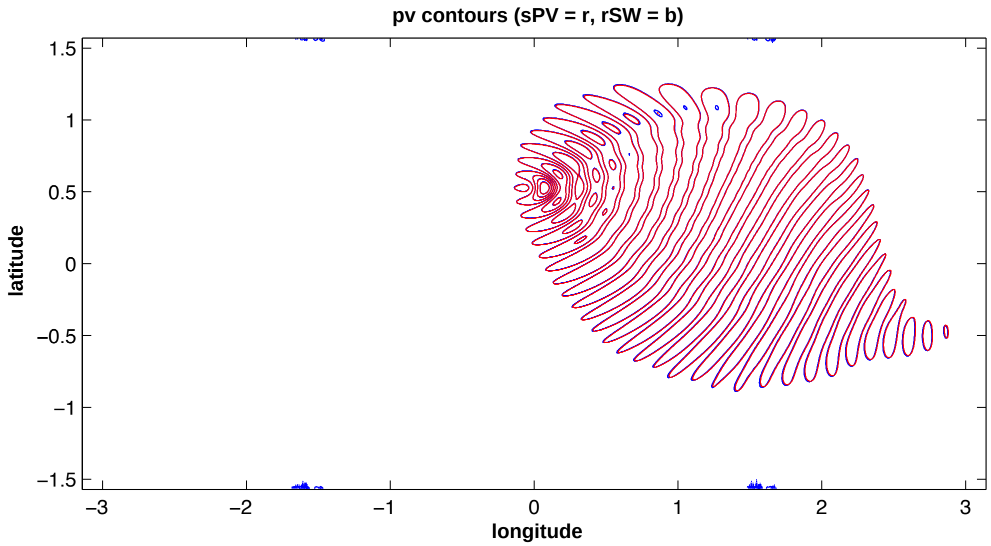
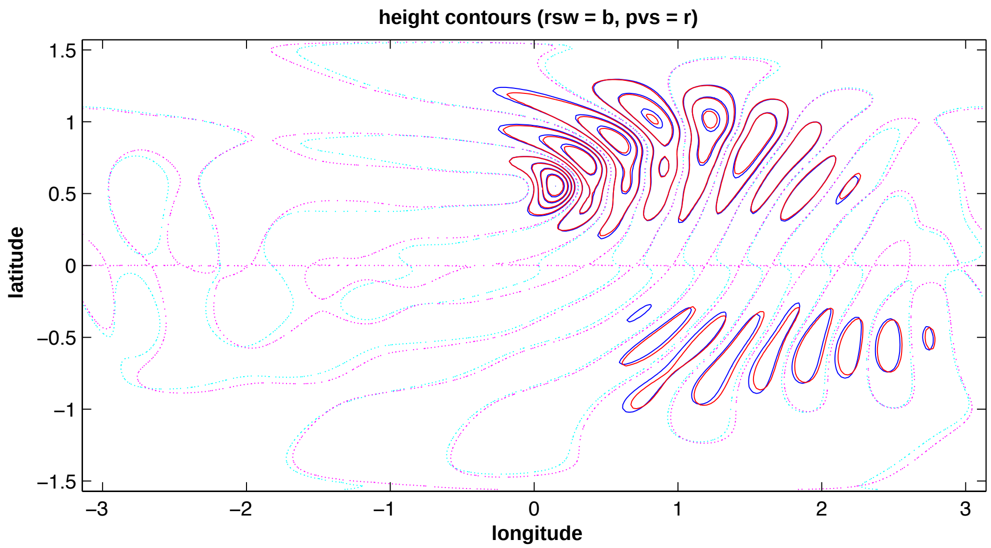
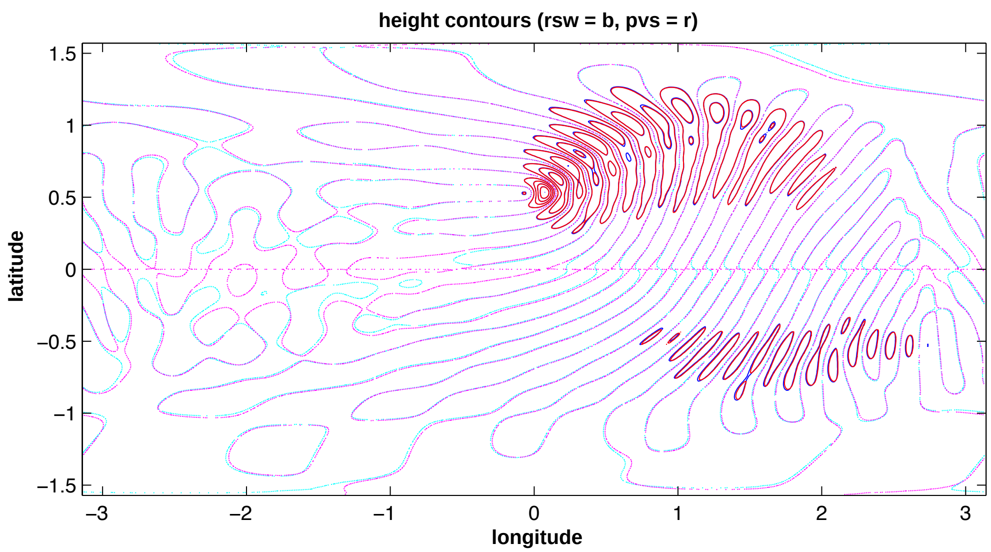




Wavepacket Limit of Short-Scale Waves: $\epsilon = \frac{1}{2}\,\epsilon_{\mathrm{gh}}$ & $\frac{1}{4}\,\epsilon_{\mathrm{gh}}$





Geometrical Optics Limit as $\epsilon \longrightarrow 0$

- (flow) = (steady, zonal $\bar{u}(\phi)$) + (wave amplitude) $e^{iS(\lambda,\phi,t)/\epsilon}$
- phase $S(\lambda, \phi, t)$ satisfies ϵ-independent Hamilton-Jacobi PDE

$$\left(S_t + \bar{u}\,\frac{S_\lambda}{\cos\phi}\right)\left(\frac{S_\lambda^2}{\cos^2\phi} + S_\phi^2 + \frac{\sin^2\phi}{1+\epsilon\,\bar{\eta}}\right) - \frac{S_\lambda}{\cos\phi}\left(\frac{\sin\phi}{1+\epsilon\,\bar{\eta}}\right)_\phi (1+\epsilon\,\bar{\eta}) = 0$$

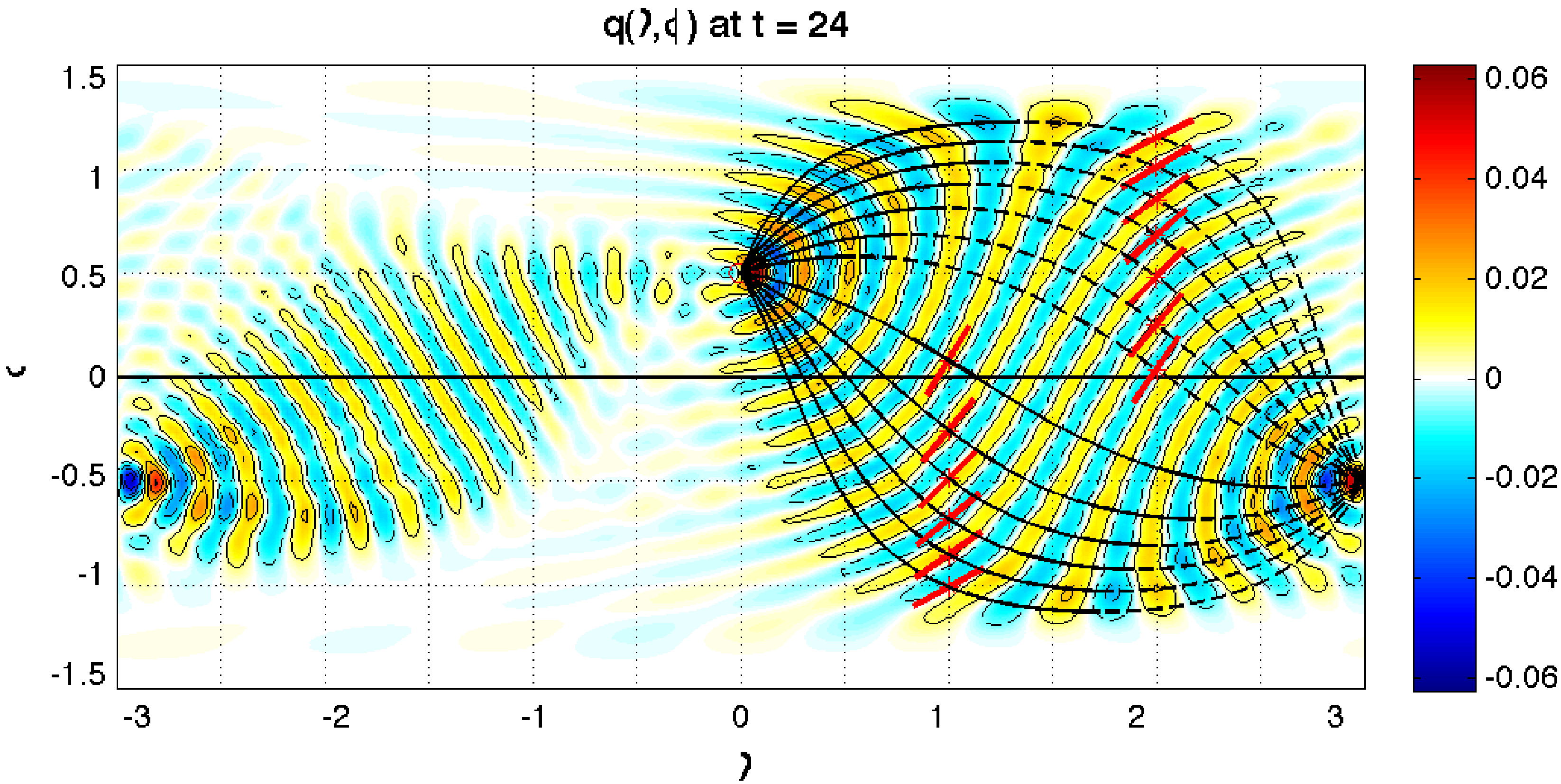
- midlatitude replacements . . .

$$S_t \longrightarrow -\omega \;\;;\;\; \frac{S_\lambda}{\cos\phi} \longrightarrow k \;\;;\;\; S_\phi \longrightarrow l \;\;;\;\; \sin\phi \longrightarrow \tilde{f} \;\;;\;\; \cos\phi \longrightarrow \tilde{\beta} \;\;;\;\; \epsilon\,\bar{\eta} \longrightarrow 0$$

- . . . give midlatitude Rossby wave dispersion (Hoskins & Karoly, 1981)

$$\omega = \bar{u}\,k - \frac{\tilde{\beta}k}{k^2 + l^2 + \tilde{f}^2}$$

Q: So, Why are G&H Equatorial Crossings Faithfully Represented?

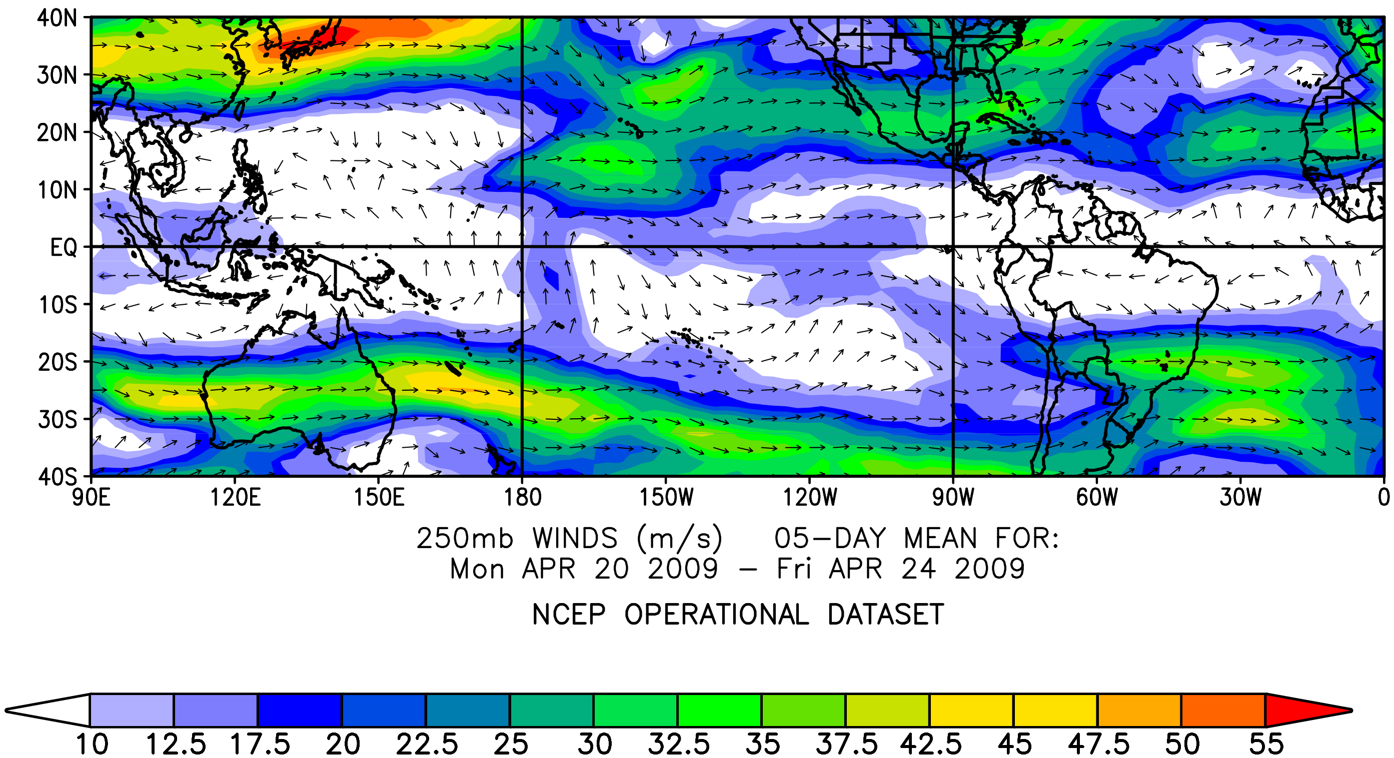


A: Both sPV & slow modes of rSW have the same global ray theory!

- ▷ sPV dynamics & steady group rays with phasefronts

NCEP Operational Analysis

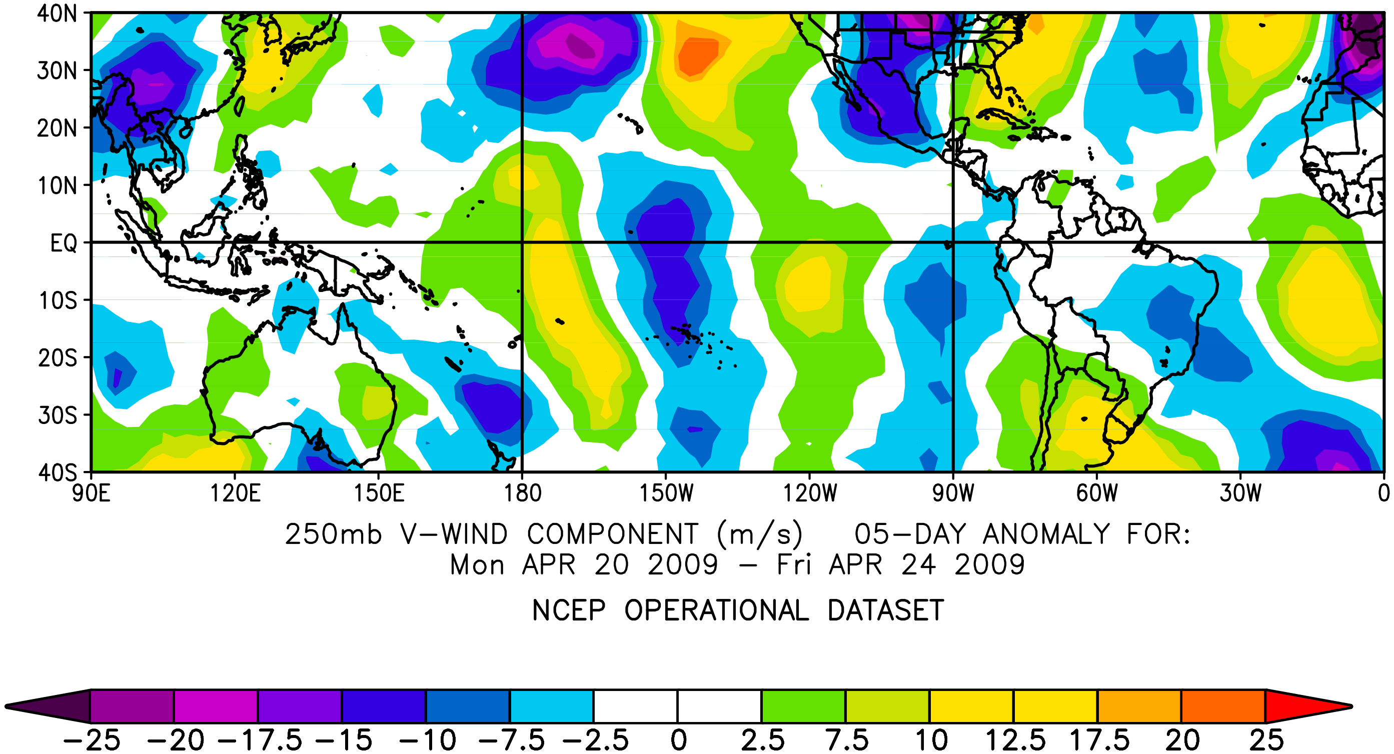
- 250mB wind vectors (5-day average) — thanks to Mel Shapiro, NCAR
- Rossby wave flow in Equatorial Pacific upper troposphere



20-24 April 2009

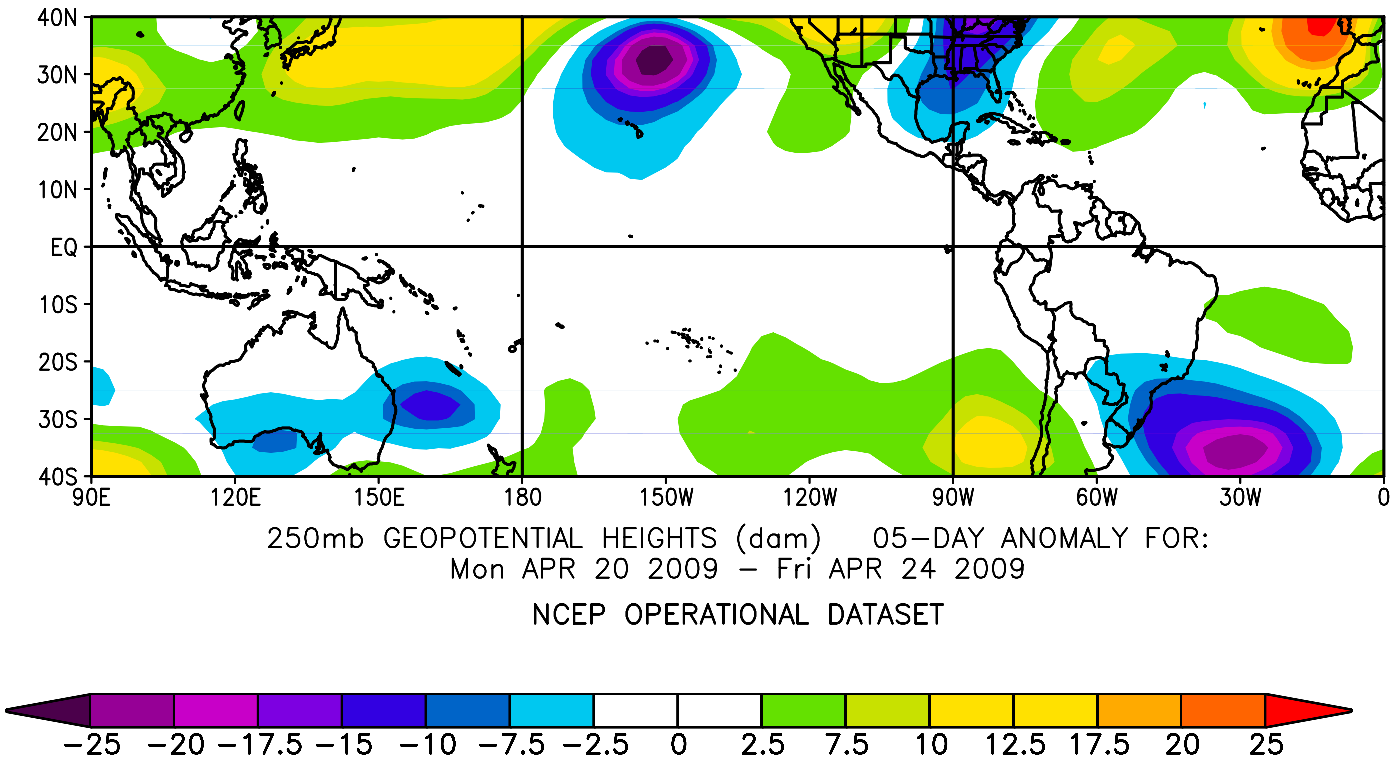
NCEP Operational Analysis

- ▷ 250mB meridional wind anomaly (5-day average)
- ▷ Rossby wave flow in Equatorial Pacific upper troposphere



NCEP Operational Analysis

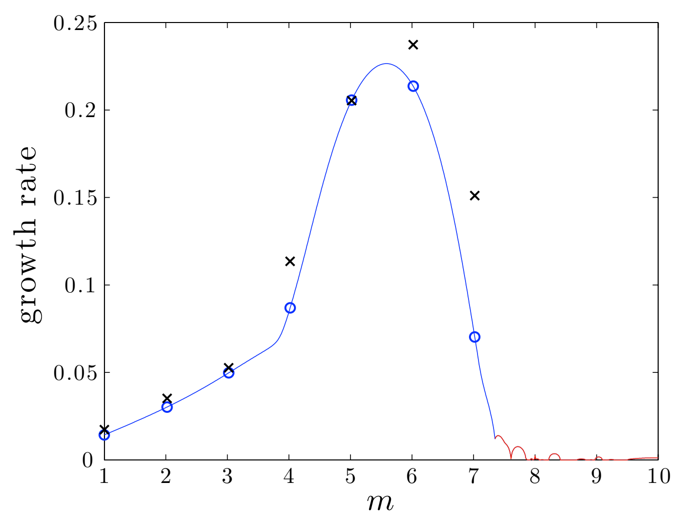
- ▷ 250mB geopotential height anomaly (5-day average)
- ▷ Rossby wave flow in Equatorial Pacific upper troposphere

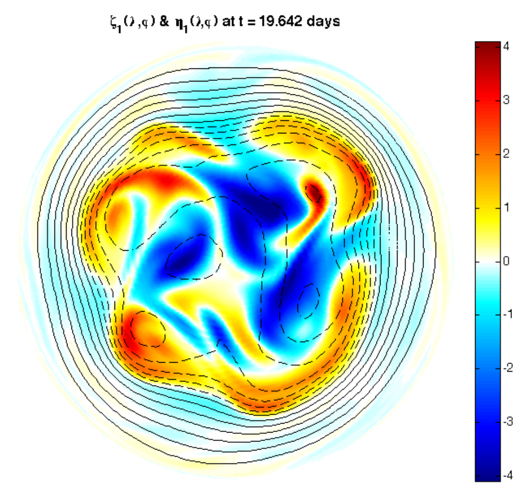


In Closing

sPV as a Balanced Dynamics for the rSW Sphere

- ▷ slow modes only
  - ▹ no gravity waves . . . but, kelvin waves . . . (next week ???)
- ▷ nonlinear advection of disturbance PV with streamfunction inversion
  - ▹ finite ϵ-theory, but is locally QG in midlatitudes
- ▷ small Rossby number limit
  - ▹ identical (linear) ray theory with (slow modes of) rSW
- ▷ recreates Grose & Hoskins (1979) equatorial wave crossing (includes large-scale jet)
- ▷ recreates Warn (1976) linearized, 2-layer baroclinic instability





2-Layer sPV Dynamics + an Unstable Jet [A. Blazenko, MSc]

PV-Streamfunction: 2 coupled elliptic equations, $q_1, q_2 \longrightarrow \psi_1,\ \psi_2 \quad (k = 1, 2)$

$$\left[\epsilon^2 \gamma_1 \nabla^2 \quad - \quad \left(\frac{1}{1-\delta}\right) \sin^2\phi\right]\psi_1 \quad + \quad \left(\frac{1}{1-\delta}\right) \sin^2\phi\ \psi_2 \quad = \quad \gamma_1^2\, q_1$$

$$\left[\epsilon^2 \gamma_2 \nabla^2 \quad - \quad \left(\frac{1}{1-\delta}\right) \sin^2\phi\right]\psi_2 \quad + \quad \left(\frac{\delta}{1-\delta}\right) \sin^2\phi\ \psi_1 \quad = \quad \gamma_2^2\, q_2$$

- fractional depths

$$\gamma_1 = \left(\frac{\alpha}{1+\alpha}\right) + \epsilon\,(\bar{\eta}_1 - \bar{\eta}_2) \quad ; \quad \gamma_2 = \left(\frac{1}{1+\alpha}\right) + \epsilon\,\bar{\eta}_2$$

sPV Balance Relations: $\psi_k \longrightarrow u_k, v_k, \eta_k$

$$u = -\epsilon \frac{\partial \psi_k}{\partial \phi} \quad ; \quad v = \frac{\epsilon}{\cos\phi} \frac{\partial \psi_k}{\partial \lambda} \quad ; \quad \eta = \psi_k \sin\phi.$$

Disturbance PV Advection: time evolution

$$\frac{DQ_k}{Dt} = \epsilon \left\{ \frac{\partial q_k}{\partial t} + \frac{\bar{u}_k + u_k}{\cos\phi} \frac{\partial q_k}{\partial \lambda} + v \frac{\partial q_k}{\partial \phi} \right\} + v_k \left(\frac{\sin\phi}{\gamma_k}\right)_\phi = 0$$

2-Layer rSW Baroclinic Instability

Linearized rSW, Warn (1976)

- identify unstable eigenvalues; expect sPV results to compare closely for small ϵ

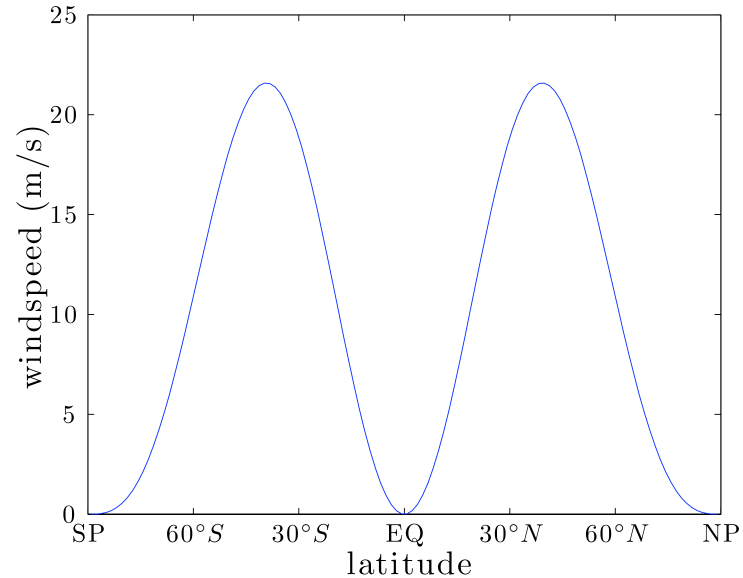
Jet Profile

- mean jet in upper layer

$$\bar{u}_1 = \frac{1}{8}\sin^2\phi\cos^3\phi$$

- max jet speed $\approx$ 22 m/s at 40°
- no flow in lower layer

$$\bar{u}_2 = 0$$



Parameter values

- $\epsilon = \sqrt{2/13} \approx 0.4$
- $\alpha \equiv H_1/H_2 = 1$
- $\delta \equiv \rho_1/\rho_2 = 0.9$

2-Layer rSW Baroclinic Instability

Linearized rSW, Warn (1976)

- identify unstable eigenvalues; expect sPV results to compare closely for small ϵ

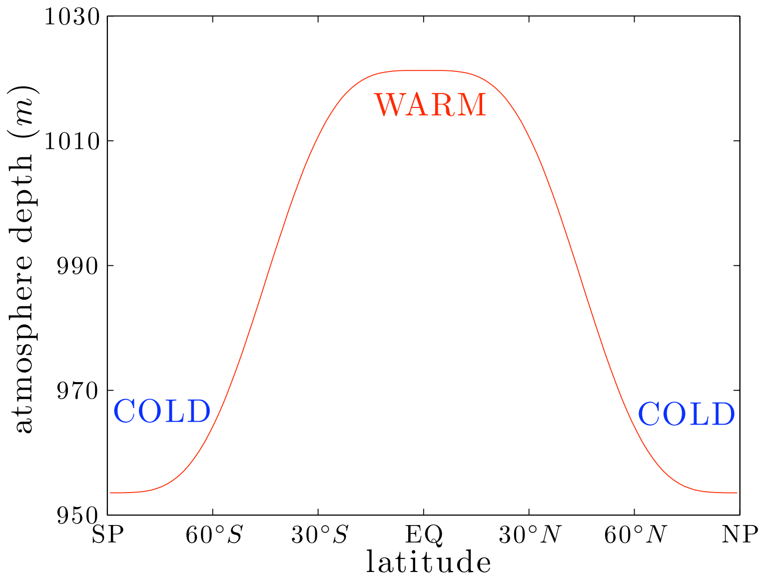
Jet Profile

- disturbance height in upper layer

$$\epsilon\,\bar{\eta}_1 = \frac{1}{8}\left(\frac{1}{6}\sin^6\phi - \frac{1}{4}\sin^4\phi + \frac{11}{420}\right)$$

$$\epsilon\,\bar{\eta}_2 = -\frac{\delta}{1-\delta}\,\epsilon\,\bar{\eta}_1$$

- mean depth: $1 + (\epsilon\,\bar{\eta}_1 - \epsilon\,\bar{\eta}_2)$



Parameter values

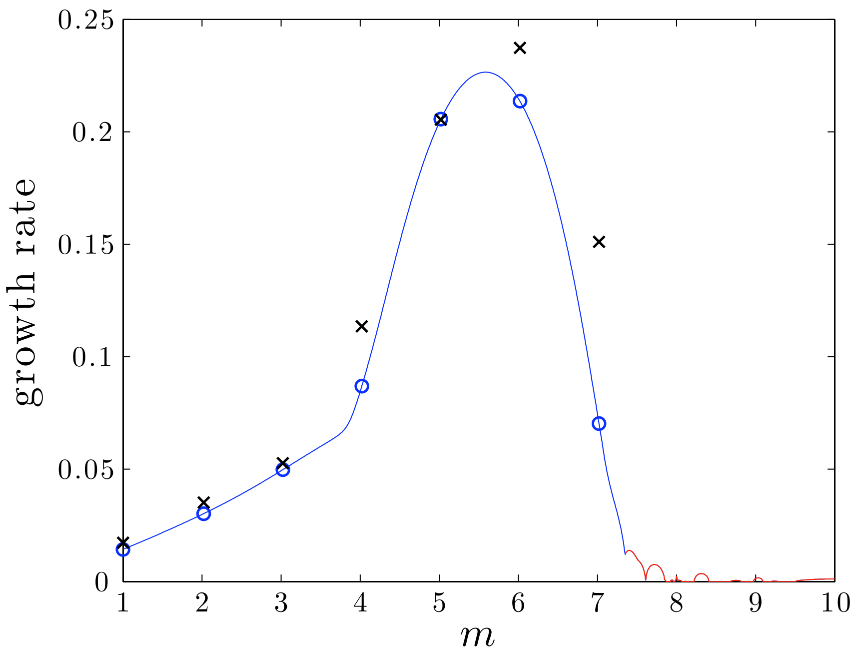
- $\epsilon = \sqrt{2/13} \approx 0.4$
- $\alpha \equiv H_1/H_2 = 1$
- $\delta \equiv \rho_1/\rho_2 = 0.9$

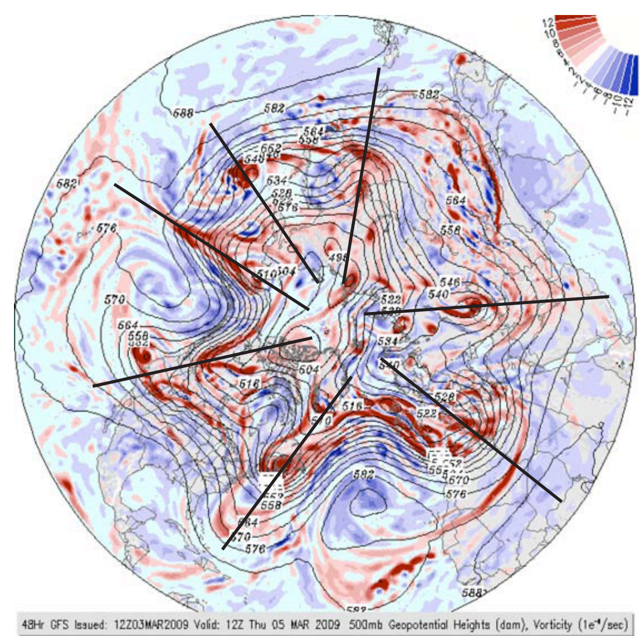
Unstable Growth Rates: $(\epsilon \approx 0.4)$

Maximum Growth Rates (day^{-1})

	m = 1	2	3	4	5	6	7
sPV	0.014404	0.029759	0.049603	0.084604	0.20412	0.21476	0.074311
rSW	0.0173	0.0352	0.0527	0.1135	0.2054	0.2373	0.1511*

- most unstable mode: wavenumber $m = 6$
- short wave cutoff [Charney (1947), Eady (1949)]

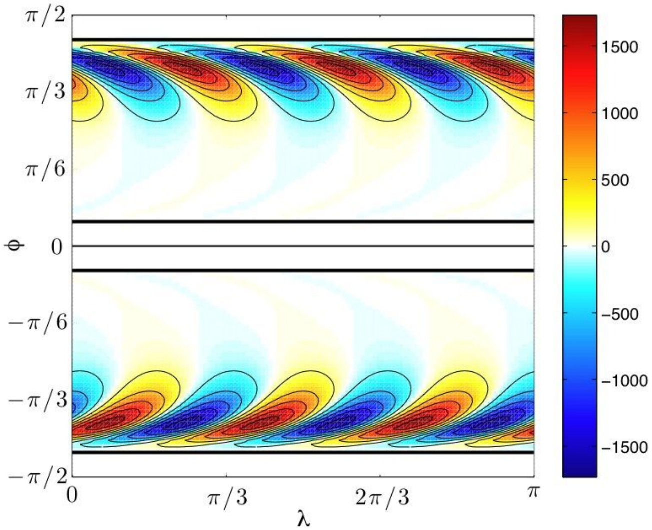




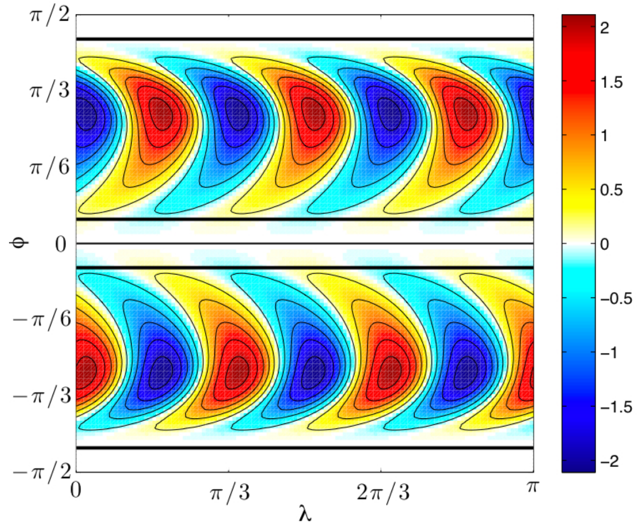
Most Unstable Mode ($m = 6$)

Upper Layer

$PV\ (q_1)$

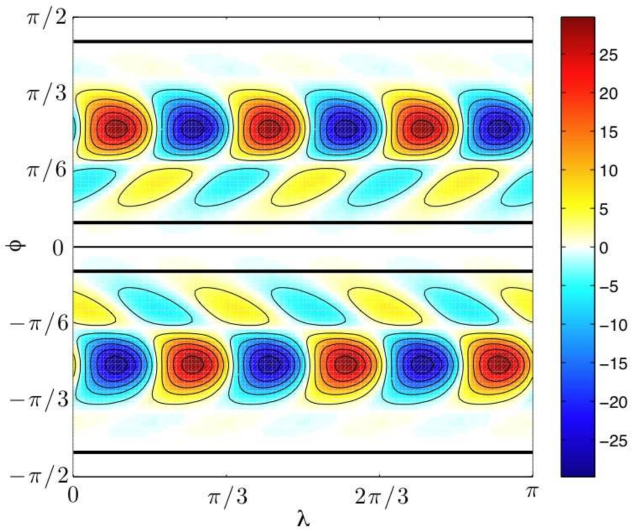


streamfunction (ψ_1)

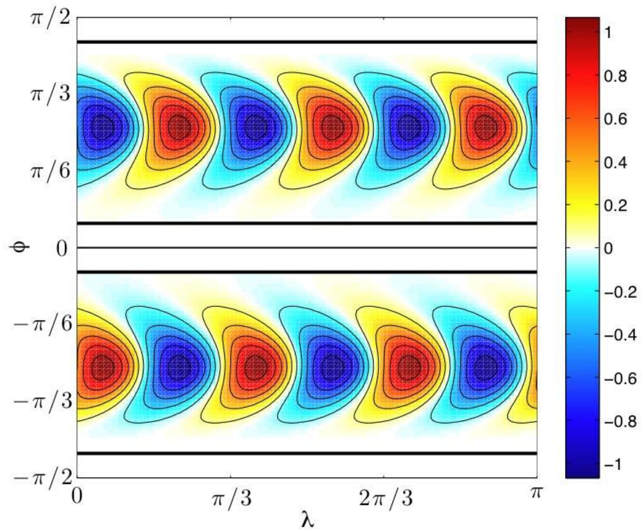


Lower Layer

$PV\ (q_2)$

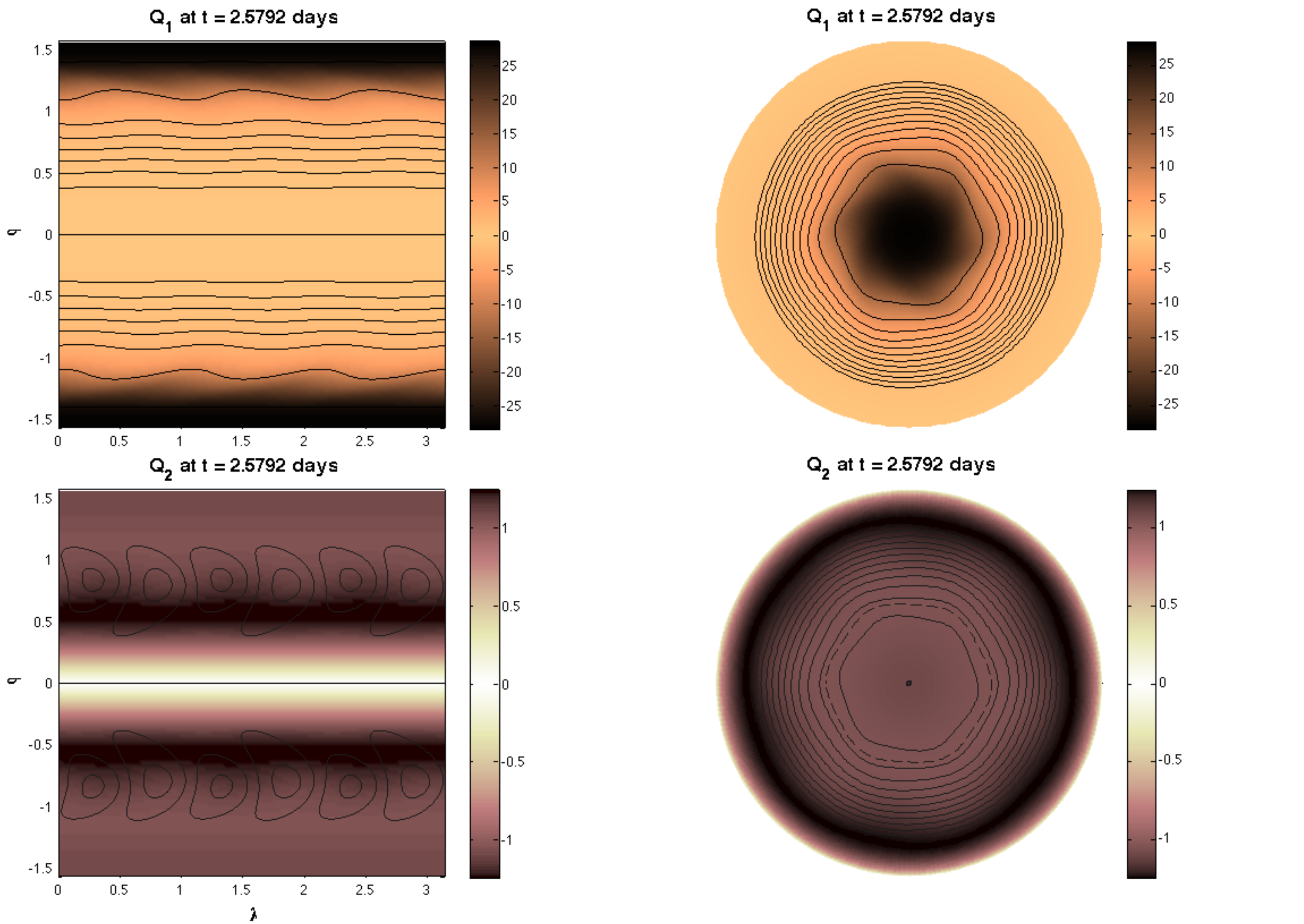


streamfunction (ψ_2)



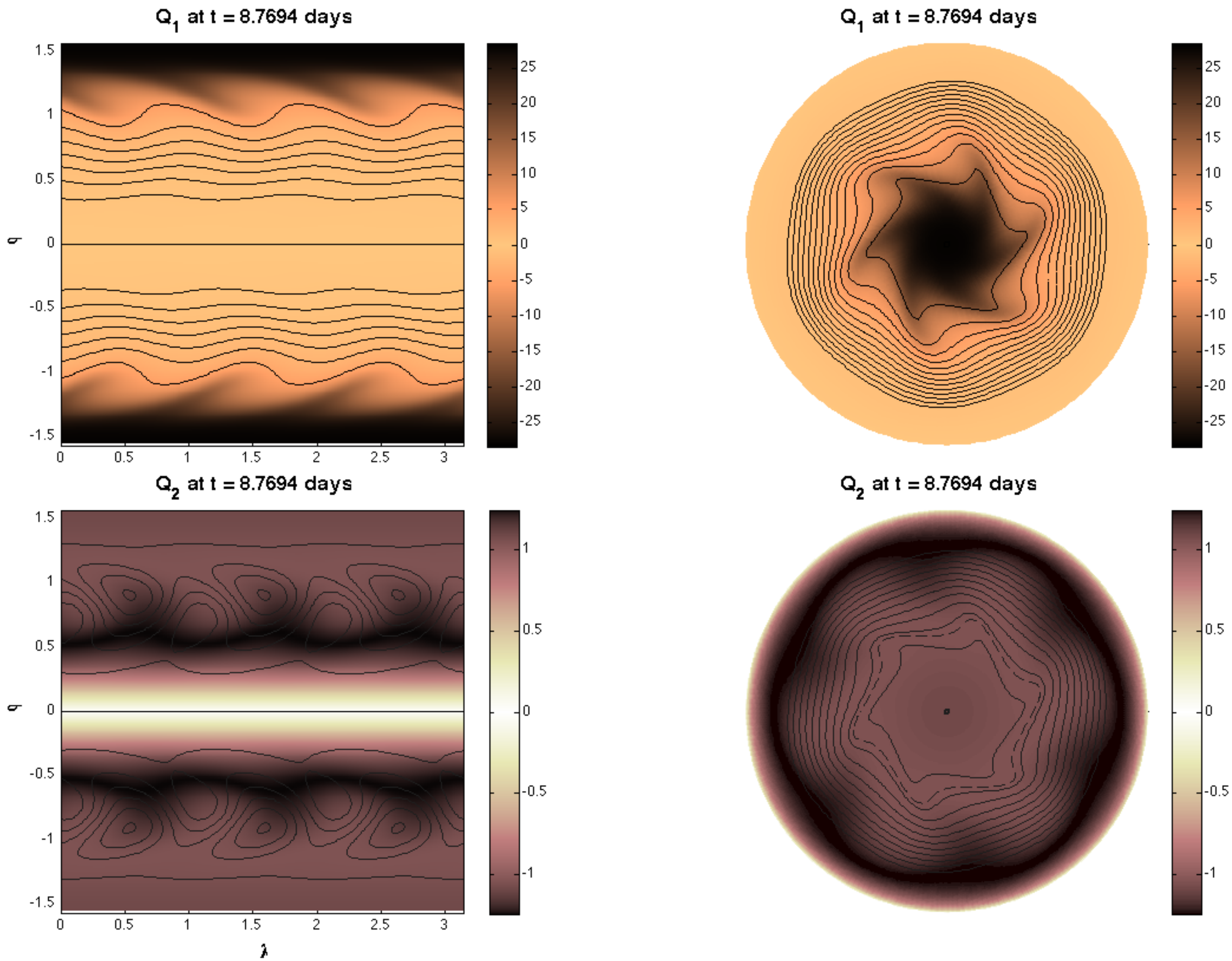
Nonlinear Evolution: $m = 6$ & $t =$ 2.58 days

Total PV & Streamfunction/Height



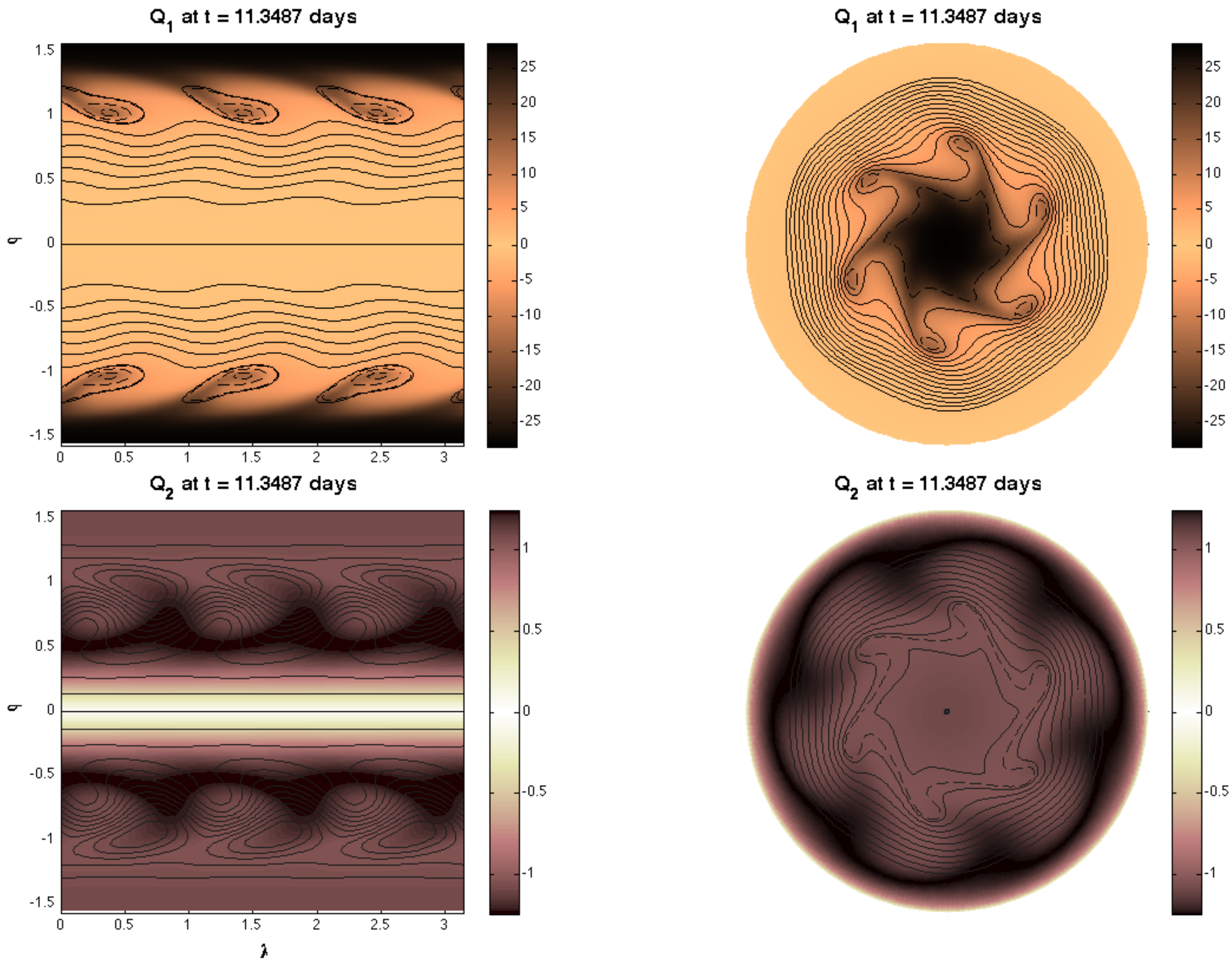
Nonlinear Evolution: $m = 6$ & $t = \; 8.77$ days

Total PV & Streamfunction/Height



Nonlinear Evolution: $m = 6$ & $t = 11.35$ days

Total PV & Streamfunction/Height



PV Dynamics for the rSW Sphere

- ▷ local asymptotic validity in midlatitudes
- ▷ accurate for global Rossby waves at scales smaller than planetary
- ▷ recreates Grose & Hoskins (1979) equatorial wave crossing
- ▷ shares identical (linear) ray theory with rSW
  - ▷ rSW ray theory is degenerate theory, but globally valid
- ▷ sPV includes nonlinear PV advection