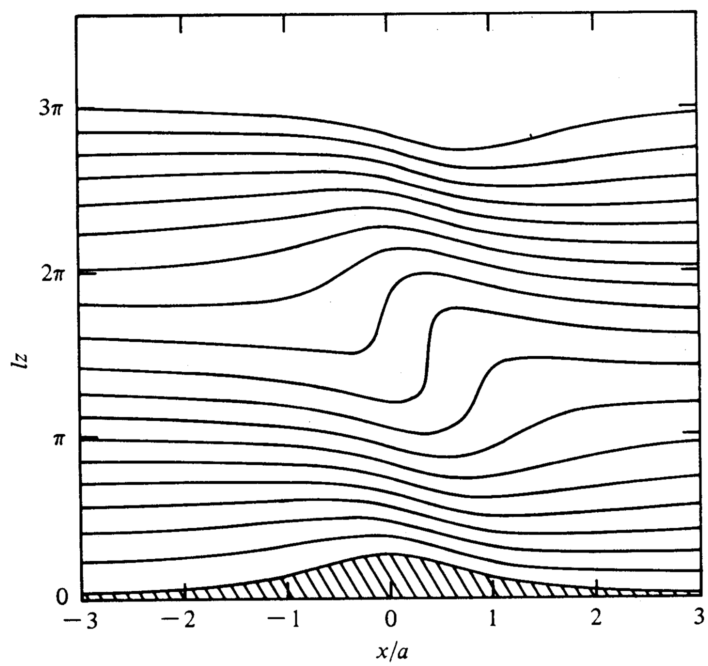
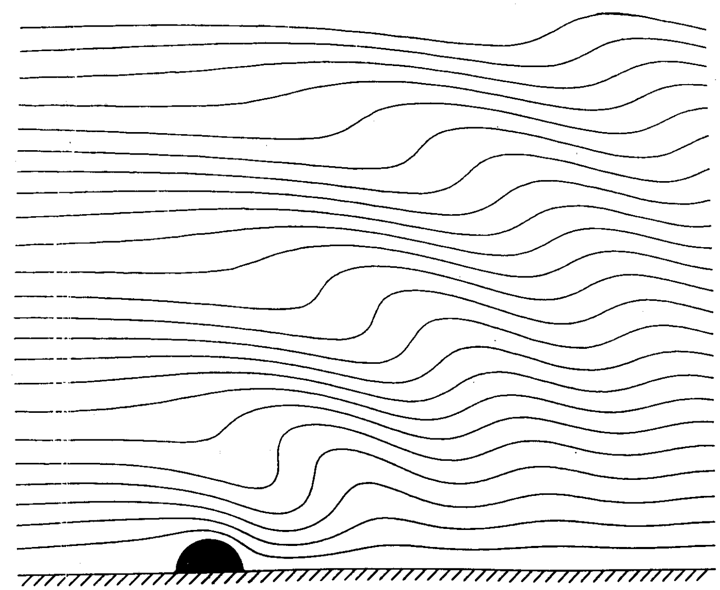
A Few Surprises in 2D Nonlinear Flow over Topography

▷ computing Long's theory with exact surface boundary conditions



Lilly/Klemp 1979



Miles/Huppert 1968

▷ Dave Muraki (Simon Fraser University)

Revisiting Long's 1953 Theory

Two-Dimensional Primitive Equations

- inviscid, incompressible, Boussinesq buoyancy

$$
\begin{aligned}
u_x \quad + \quad w_z &= 0 \\
\frac{Du}{Dt} \qquad\qquad &= -\phi_x \\
\delta^2 \frac{Dw}{Dt} \quad - \quad \theta &= -\phi_z \\
\frac{D\theta}{Dt} \quad + \quad w &= 0
\end{aligned}
$$

- nonhydrostatic parameter ($\delta = U/NL$) & height scale ($\mathcal{A} = NH/U$)
- potential temperature (θ) & geopotential (ϕ)

Steady Streamfunction: $\psi(x, z) = z + \tilde{\psi}(x, z)$

- uniform upstream wind U & constant stratification N
- exact reduction to <u>linear</u> Helmholtz equation for disturbance streamfunction

$$\delta^2 \, \tilde{\psi}_{xx} \; + \; \tilde{\psi}_{zz} \; + \; \tilde{\psi} \; = \; 0$$

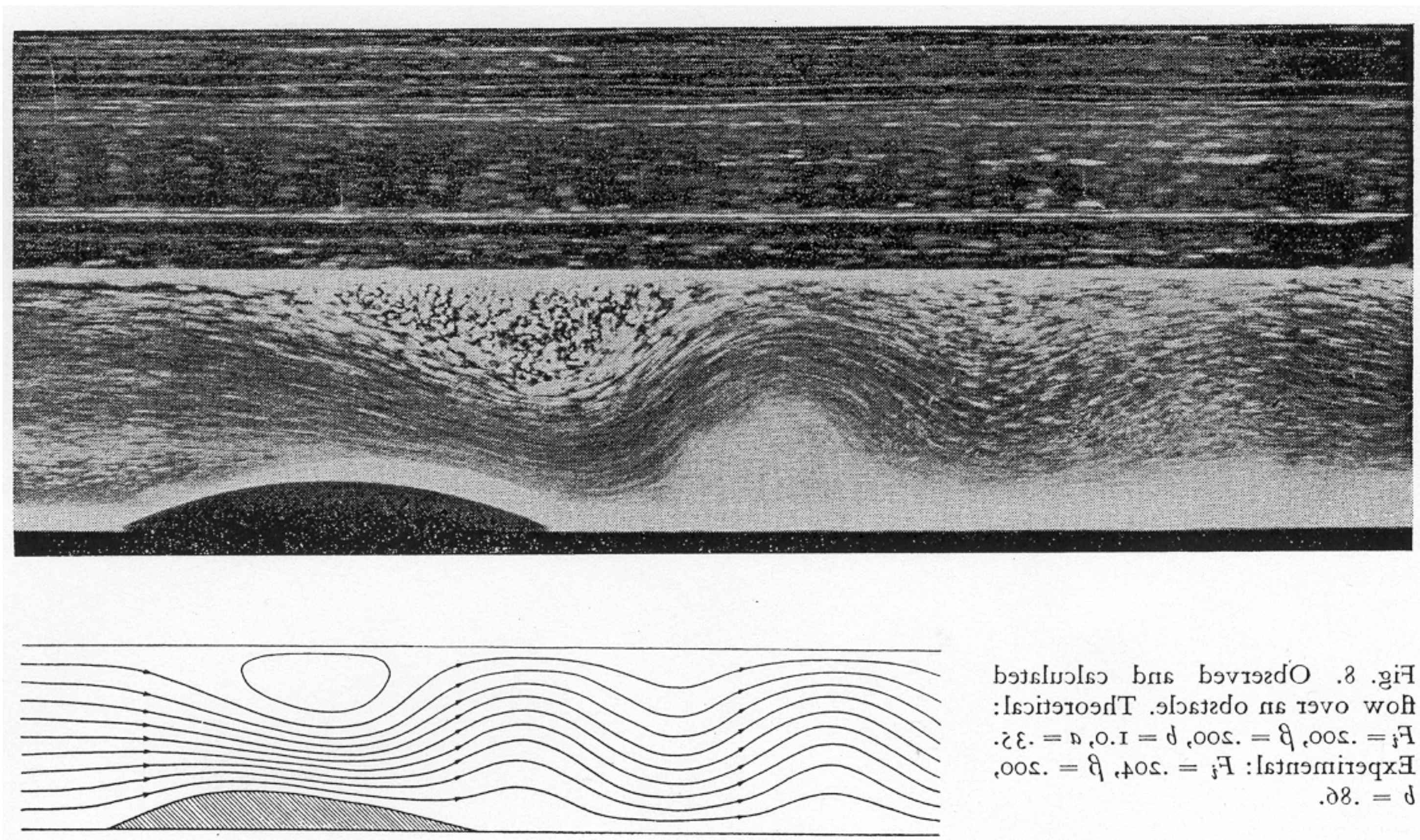
- topographic surface at $z = \mathcal{A}h(x)$ & streamline condition $\rightarrow$ $\psi(x, \mathcal{A}h(x)) = 0$
- radiation/decay BCs aloft

Long 1955: Theory & Experiment

$$\delta^2 \tilde{\psi}_{xx} + \tilde{\psi}_{zz} + \tilde{\psi} = 0$$

Finite Amplitude Topography

- on streamline boundaries: $\psi = \mathcal{A}h(x) + \tilde{\psi}(x, \mathcal{A}h(x)) =$ constant



Long 1953

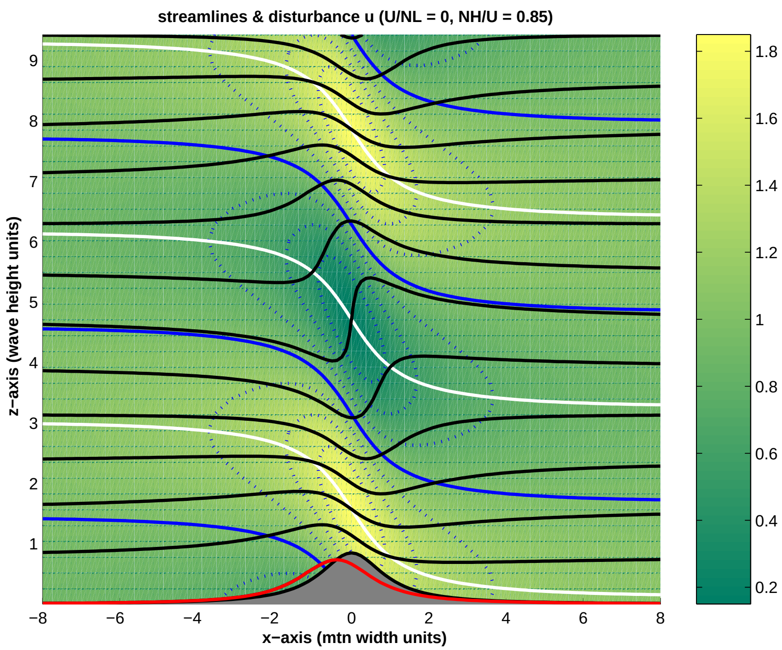
Linearized Surface Condition

$$\tilde{\psi}(x,z) \;=\; -\frac{\mathcal{A}}{2\pi}\int_{-\infty}^{+\infty} \hat{h}(k)\; e^{i(kx+m(k)z)}\; dk$$

Fourier Solution (for small $\mathcal{A}$)

- ▷ boundary at $z = 0$ & <u>linearized</u> topographic condition $\longrightarrow$ $\mathcal{A}h(x) + \tilde{\psi}(x,0) = 0$
- ▷ aloft conditions via vertical mode number ($\delta^2 k^2 + m^2 = 1$)

$$m(k) = \begin{cases} \operatorname{sign}(k)\,\sqrt{1-\delta^2 k^2} & \text{for } |\delta k| \le 1 \text{ (long scale radiation)} \\ i\,\sqrt{\delta^2 k^2 - 1} & \text{for } |\delta k| \ge 1 \text{ (short scale decay)} \end{cases}$$



General Helmholtz Solution

$$\tilde{\psi}(x,z) \;=\; -\mathcal{A}\int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)z)}\; dk$$

Fourier Representation

- satisfies aloft conditions ($\delta^2 k^2 + m^2 = 1$)
- surface at $z = \mathcal{A}h(x)$ & exact topographic condition $\;\longrightarrow\;$ $\mathcal{A}h(x) + \tilde{\psi}(x, \mathcal{A}h(x)) = 0$

$$h(x) \;-\; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)\mathcal{A}h(x))}\; dk \;=\; 0$$

Fredholm Integral Equation of the First-Kind

- linearity: action of integral operator is linear in unknown coefficients $\hat{c}(k)$
- numerical solution equivalent to matrix inversion
- velocities from spectral differentiation: $u = \psi_z$ & $w = -\psi_x$
- no need to compute Fourier transform: $c(x) \longrightarrow$ *effective topography*

Direct Steady Solve

$$h(x) \;-\; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)\mathcal{A}h(x))}\; dk \;=\; 0$$

Numerical Discretization

- collocation points: $\{x_1 \;...\; x_\alpha \;...\; x_N\}$ & N knowns: $h_\alpha = h(x_\alpha)$
- wavenumbers: $\{k_1 \;...\; k_\beta \;...\; k_N\}$ & N unknowns: $\hat{c}_\beta \approx \hat{c}(k_\beta)$
- approximate integral at each x_α by quadrature (trapezoidal rule) over $\beta = 1 \ldots N$

$$h_\alpha \;-\; \sum_{\beta=1}^{N} \hat{c}_\beta \; \underbrace{e^{i(k_\beta x_\alpha+m(k_\beta)\mathcal{A}h(x_\alpha))}\; w_\beta\, \Delta k}_{\mathbf{K}_{\alpha,\beta}} \;=\; 0$$

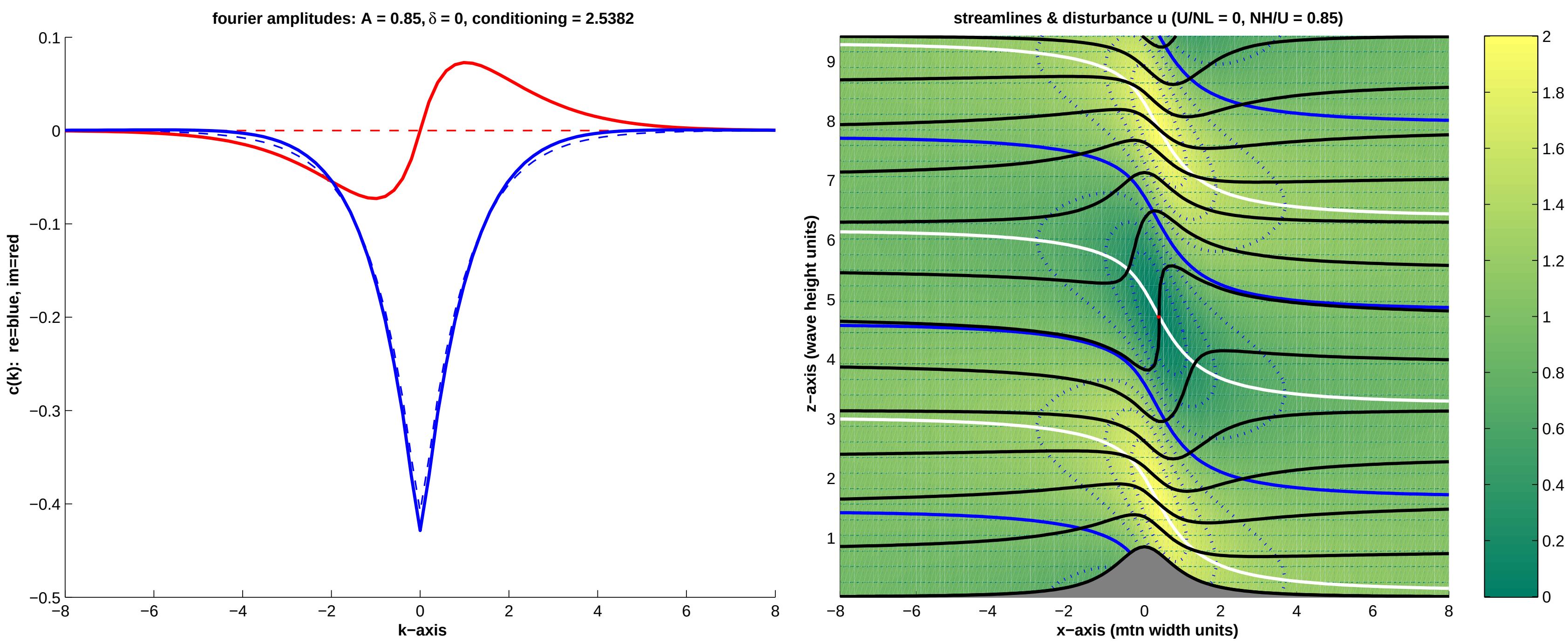
Matrix Inversion

- N linear equations in N unknowns: $(\vec{h_\alpha}) = [\mathbf{K}_{\alpha,\beta}]\;(\vec{c_\beta})$
- $m(k)$ is discontinuous at $k = 0 \longrightarrow$ half-line integrals
- full matrix $\mathbf{K}$ can be ill-conditioned $\longrightarrow$ catastrophic loss of precision as N increases

Numerical Implementation

Fourier Conditioning

- ▷ $\mathcal{A} = 0$ recovers linear theory & discrete Fourier transform is well-conditioned
- ▷ equi-spaced discretizations with $\Delta k \, \Delta x = 2\pi / N$ is essential

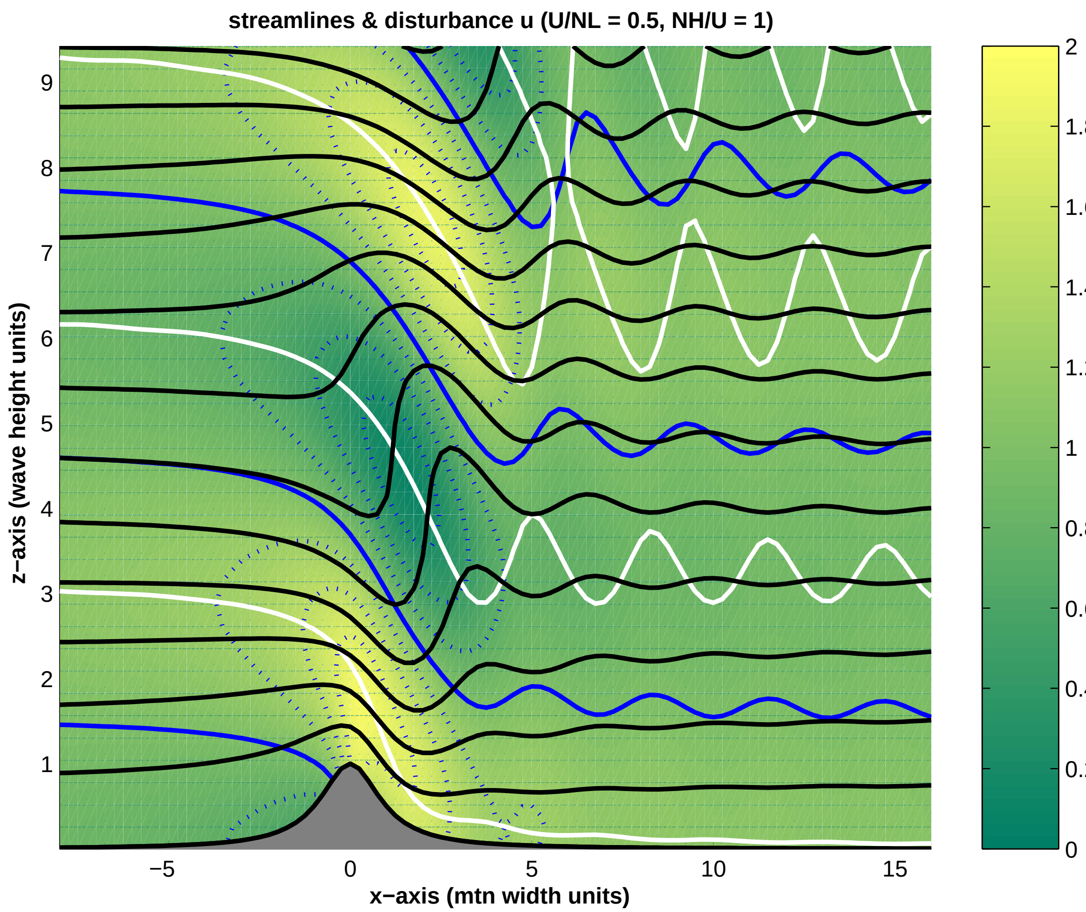


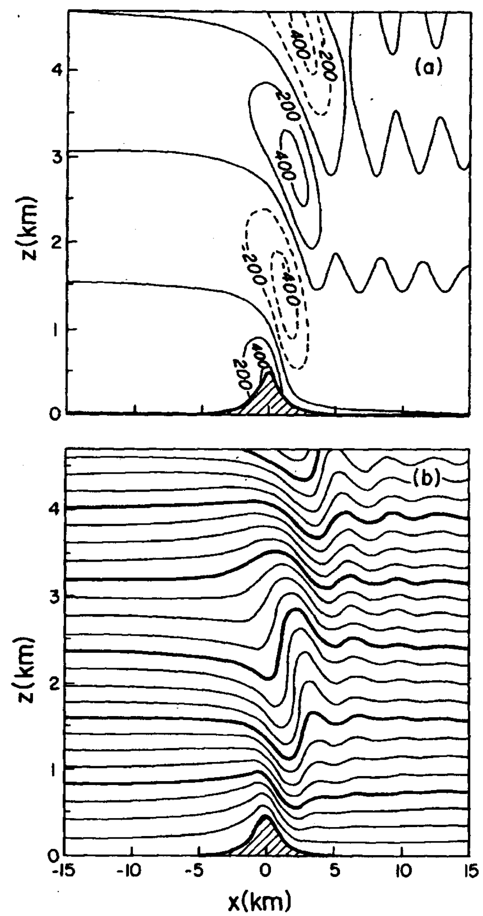
- ▷ Lilly/Klemp 1979, hydrostatic critical overturning: $N = 256 \longrightarrow$ 1.1s to solve & 2.0s to plot
- ▷ Fourier representation allows periodic wrap-around $\longrightarrow$ large computational domains

A Nonhydrostatic Example

Laprise & Peltier, 1988

- ▷ predictor/corrector to obtain effective topography $c(x) \longrightarrow$ typically 50 iterations





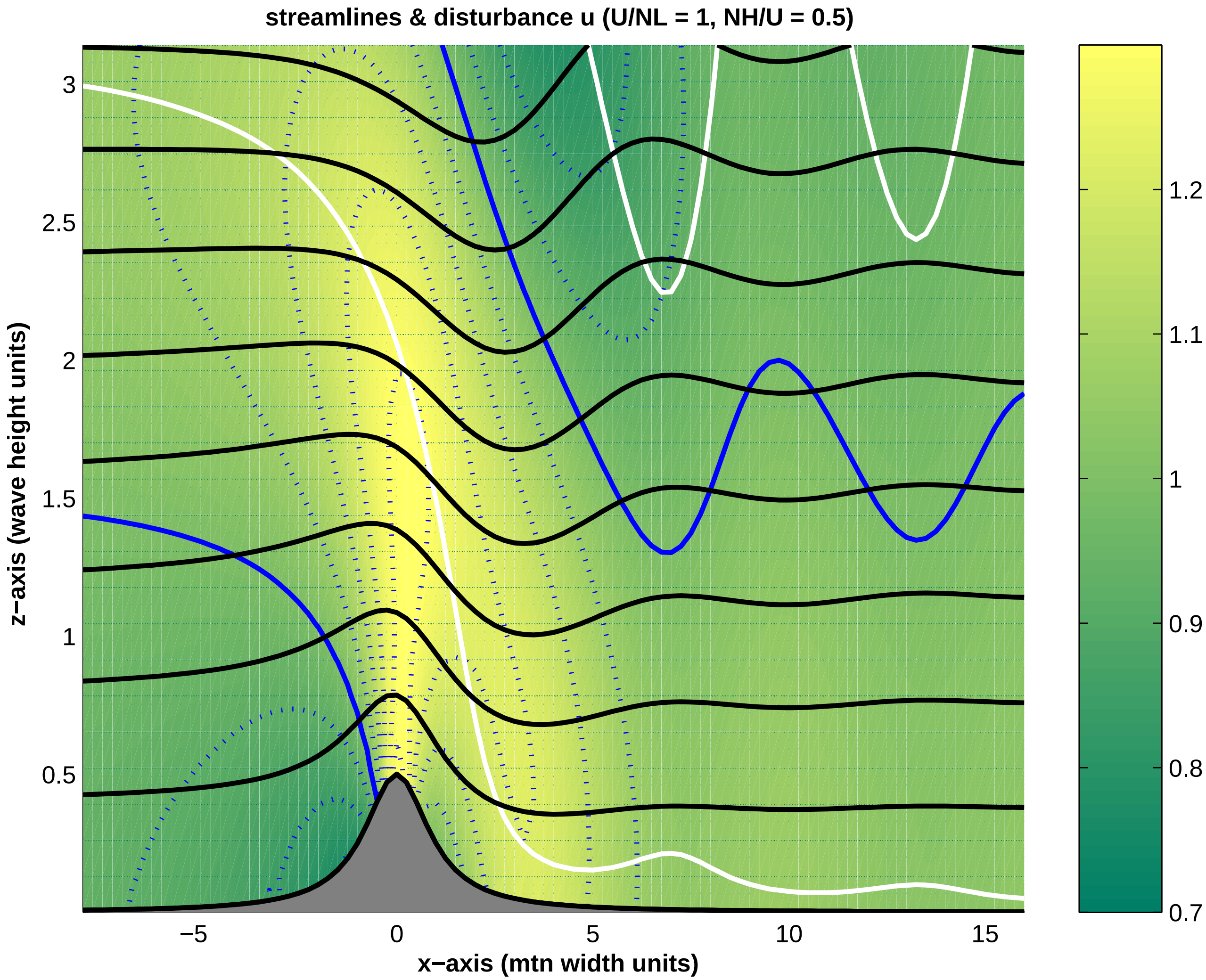
Laprise/Peltier 1988

- ▷ large amplitude $\mathcal{A} = 1.0$ & moderately nonhydrostatic $\delta = 0.5$
- ▷ $N = 2056$, $x_\infty = 256$: 284s to solve, 89s to plot, log-condition number = 5.75

A Strongly Nonhydrostatic Example

$\delta = 1.0$ & $\mathcal{A} = 0.5$

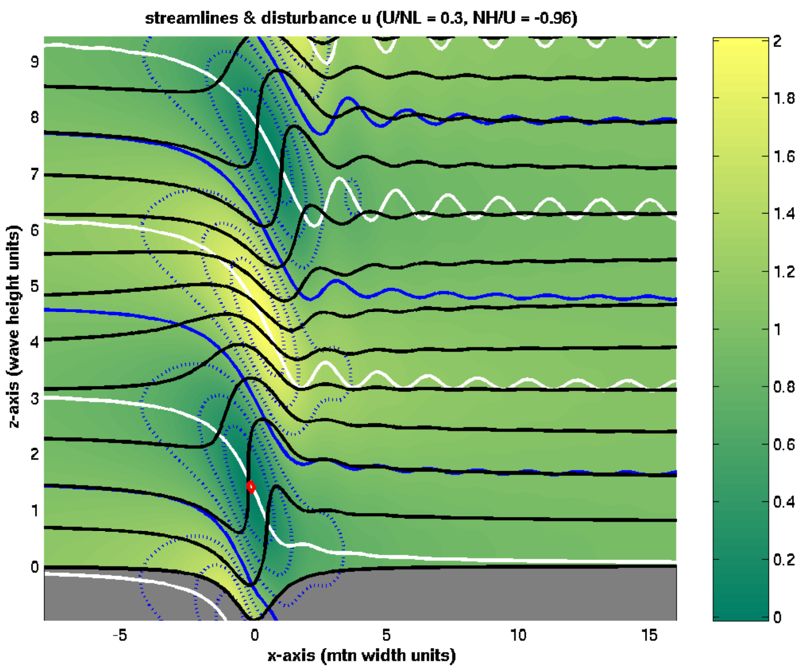
- ▷ u-wind maximum shifts towards the summit as nonhydrostatic effect increases

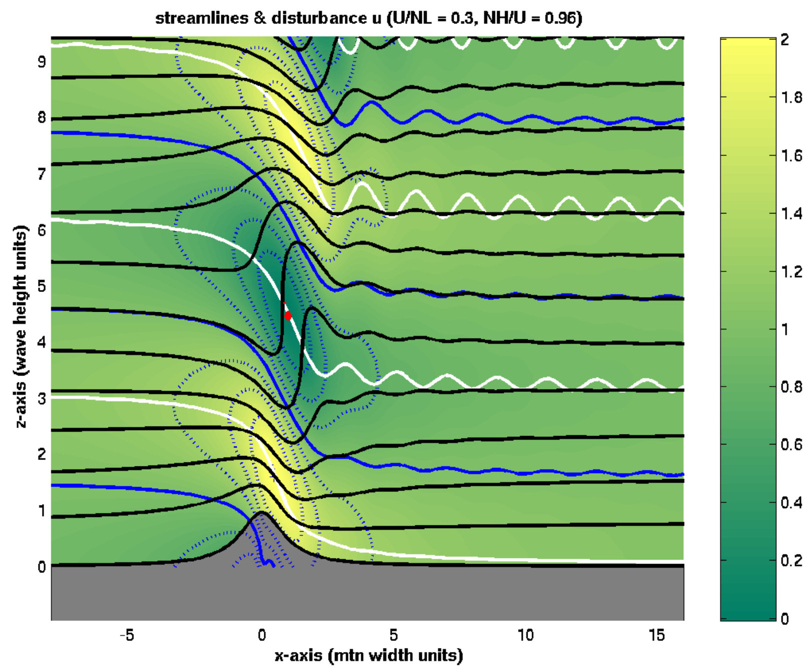


Mountain vs Valley

$\delta = 0.3$ & $\mathcal{A} = \pm 0.96$

- both cases near critical overturning
- very little asymmetry in overall magnitude of response (unlike rotating case)

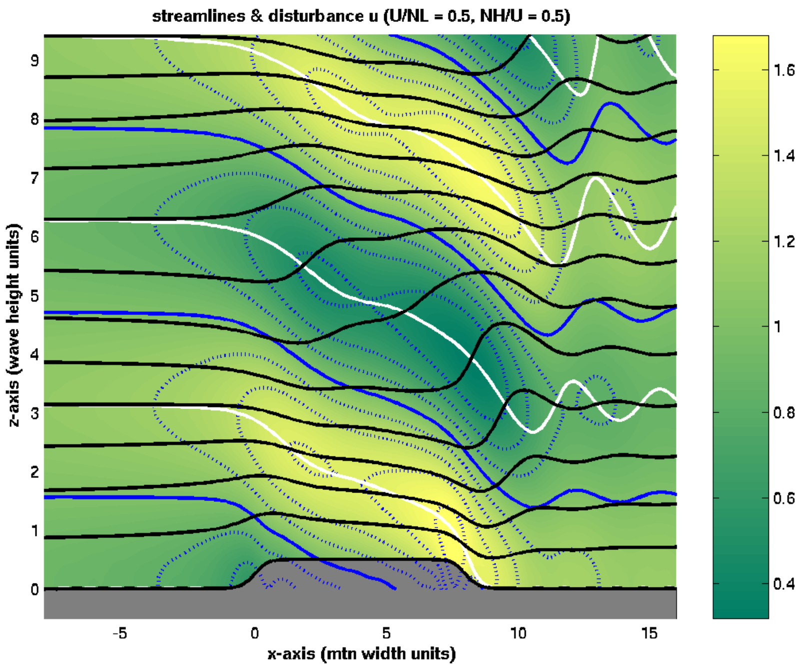


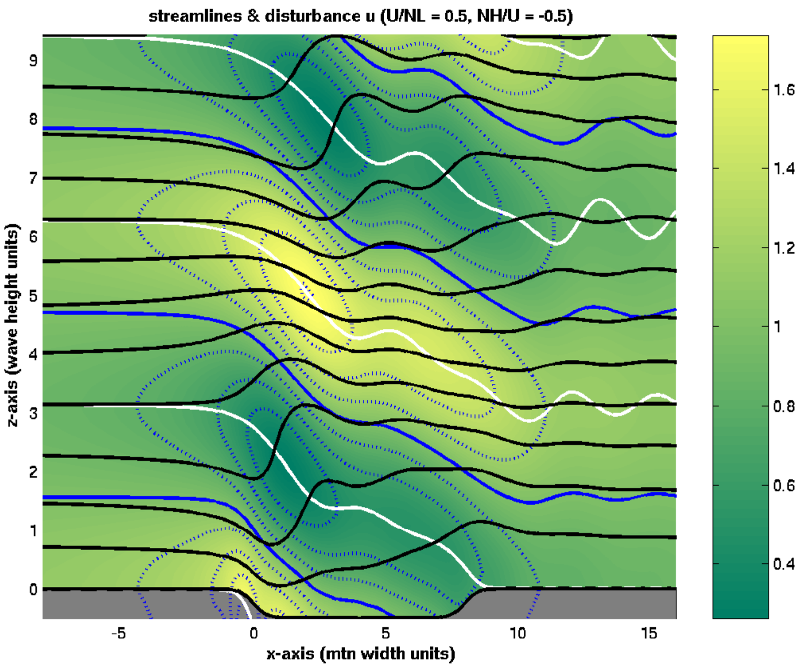


Extended Topography

$\delta = 0.5$ & $\mathcal{A} = \pm 0.5$

- largest response associated with downslope
- slightly more wind in valley case: $0.32 < u^{+} < 1.77$ vs $0.26 < u^{-} < 1.82$





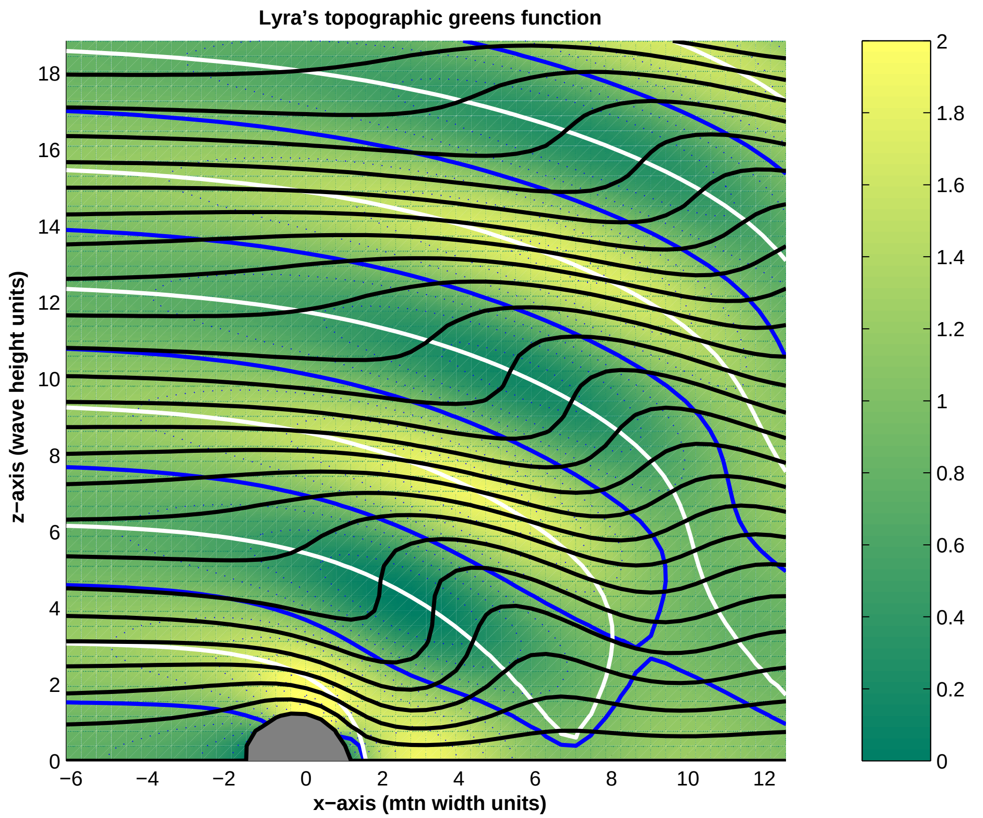
Lyra's Topographic Greens Function

Delta Function Topography

- from Lyra 1940 & 1943 (via Alaka 1960) for $\delta = 1$: Bessel series

$$\tilde{\psi}(r,\theta) \;=\; \frac{1}{2}\,Y_1(r)\sin\theta \;+\; \frac{1}{\pi}\sum_{1}^{\infty}\frac{4n}{4n^2-1}\,J_{2n}(r)\sin 2n\theta$$

- critical overturning for delta strength ≈ 4.06



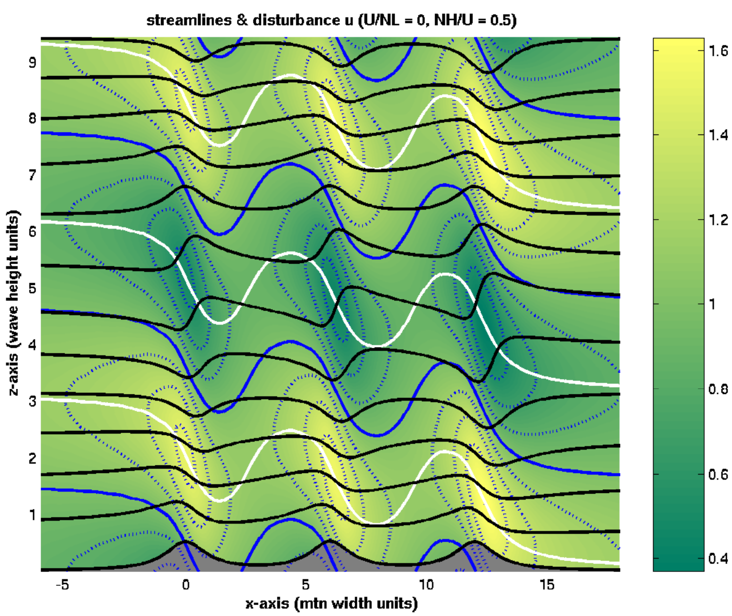
Topographic Boundary Conditions

Direct Computation

- ▷ consistent with spectral radiation condition
- ▷ elementary formulation for non-iterative solve
- ▷ Long's theory: hydrostatic & nonhydrostatic

- ▷ Fredholm first-kind integral equation is generally ill-conditioned

  $\longrightarrow$ possible resolution via Lyra's greens function
- ▷ Fourier representation allows for wrap-around of waves



- ▷ open issues in stability of Long's solutions?