Unravelling the Resonant Instabilities of a Stratified Gravity Wave

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Spectrum of Resonant Instabilities

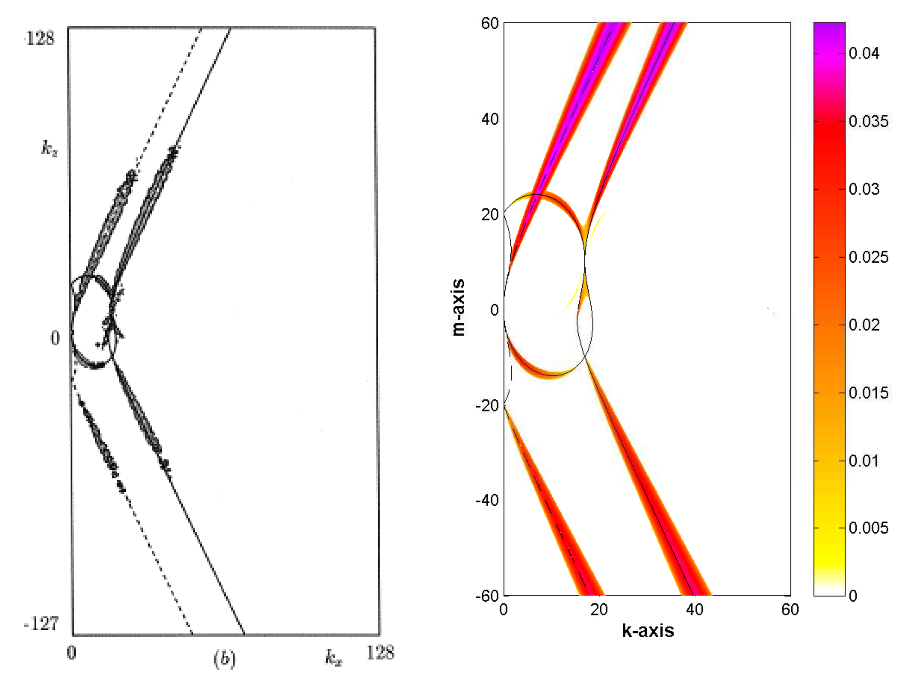

Figure 1: Spectrum of Resonant Instabilities: DNS (Lin2000) vs Floquet Unravelled Spectrum (djm & ybb)

Equations for a Stratified Fluid

2D incompressible Euler with Boussinesq Buoyancy & Constant Stratification

$$\frac{D\eta}{Dt} = -b_x \; ; \; \frac{Db}{Dt} = -w \; ; \; \nabla\cdot\vec{\mathbf{u}} = 0$$

- Buoyancy $b(x,z,t)$ and vorticity $\eta(x,z,t)$
- 2D velocity (x,z components): $\vec{\mathbf{u}} = (u,w)$
  Streamfunction $\psi(x,z,t)$: $u = \psi_z$, $w = -\psi_x$
- Advection from Jacobian:

$$J(f,\psi) = \begin{vmatrix} f_x & \psi_x \\ f_z & \psi_z \end{vmatrix} = uf_x + wf_z$$

- Vorticity: $\eta = \psi_{zz} + \delta^2\psi_{xx}$;
  Hydrostatic limit: $\delta \to 0$; Laplacian: $\delta \to 1$

Streamfunction Formulation

$$\begin{array}{rcl} \eta_t + b_x + J(\eta,\psi) &=& 0 \\ b_t - \psi_x + J(b,\psi) &=& 0 \\ \psi_{zz} + \delta^2\psi_{xx} &=& \eta \end{array}$$

Exact Nonlinear Wave Solutions

$$\begin{pmatrix}\psi\\ b\end{pmatrix} = \begin{pmatrix}-\Omega\\ K\end{pmatrix} 2\epsilon\sin(Kx+Mz-\Omega t)$$

- Primay wavenumbers: (K,M)
- Linear dispersion relation: $\Omega^2(K,M) = \dfrac{K^2}{M^2+\delta^2K^2}$

Linearized Equations

$$\begin{array}{l} \tilde\eta_t + \tilde b_x - 2\epsilon J\big(\Omega\tilde\eta + (K^2/\Omega)\tilde\psi\;,\; \sin(Kx+Mz-\Omega t)\big) = 0 \\ \tilde b_t - \tilde\psi_x - 2\epsilon J\big(\Omega\tilde b + K\tilde\psi\;,\; \sin(Kx+Mz-\Omega t)\big) = 0 \end{array}$$

- Goal: to characterize the linear instabilities of a primary wave
- Linearize w.r.t the nonlinear wave

$$\begin{pmatrix}\psi\\ b\end{pmatrix} = \begin{pmatrix}-\Omega\\ K\end{pmatrix} 2\epsilon\sin(Kx+Mz-\omega t) + \begin{pmatrix}\tilde\psi(x,z,t)\\ \tilde b(x,z,t)\end{pmatrix}$$

- Linear PDEs with periodic, non-constant coefficients
- A problem for Floquet Theory

Instabilities via Floquet Theory

Mathieu Equation

$$\ddot u + (\alpha + \epsilon\sin t)u = 0$$

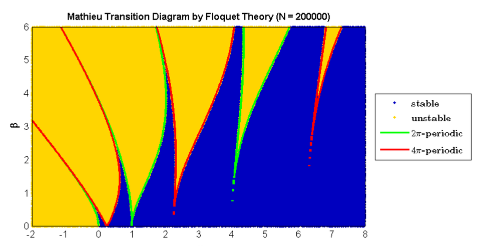

Figure 2: Spectrum of Mathieu Instabilities

- Floquet theory:

$$\vec{\mathbf{u}}(t) = e^{i\omega t}\left\{\sum_{-\infty}^{+\infty}\vec v_n e^{int}\right\} = \text{exponential part} \times \text{co-periodic part}$$

- $\omega(\alpha;\epsilon)$, Floquet exponent with Im $\omega > 0 \to$ instability.

Floquet Fourier Analysis for PDEs

- Product of exponential & co-periodic Fourier series

$$\begin{pmatrix}\tilde\psi\\ \tilde b\end{pmatrix} = e^{i(kx+mz-\omega t)}\left\{\sum_{-\infty}^{+\infty}\vec v_n e^{in(Kx+Mz-\Omega t)}\right\}$$

- Secondary/perturbed wavenumbers: (k,m)
- Floquet exponent Im $\omega(k,m;\epsilon) > 0 \to$ instability
- Hill's infinite matrix & generalized eigenvalue problem

$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix} - \omega \begin{bmatrix} \ddots & & \\ & \mathbf{\Lambda}_0 & \\ & & \mathbf{\Lambda}_1 \\ & & & \ddots \end{bmatrix}$$

- 2×2 real blocks: $\mathbf{M}_n(k,m)$; $\mathbf{S}_n(k,m)$ symmetric ; $\mathbf{\Lambda}_n(k,m)$ diagonal
- Truncate $-N \le n \le N$ & compute $4N+2$ eigenvalues $\{\omega(k,m;\epsilon)\}$

Unravelling the Spectrum

- Choose primary wavenumbers $(K,M) = (1,1)$;
  finite wave amplitude: $\epsilon = 0.1$; hydrostatic: $\delta = 0$

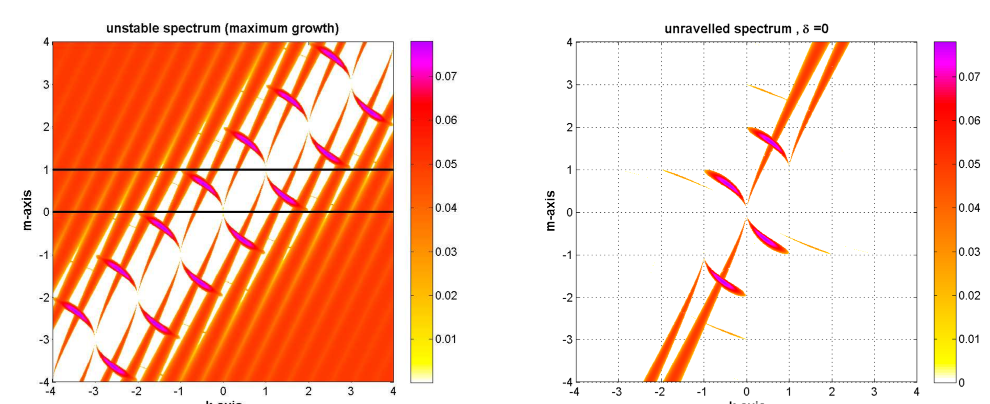

Figure 3: Raw Floquet spectrum vs unravelled Floquet spectrum

- Artificial periodicity due to index shifts $\to$ multiple counting

$$\begin{pmatrix}\tilde\psi\\ \tilde b\end{pmatrix} = e^{i((k+q)x+(m+q)z-(\omega+\Omega q)t)}\left\{\sum_{-\infty}^{+\infty}\vec v_{n+q} e^{in(x+z-\Omega t)}\right\}$$

- Resolution: to associate $\omega(k,m)$ with the instabilities given by its corresponding physical wave resonance.
- QUESTION 1. Which ω's from computation correspond to the instabilities given physical wave resonance theory ?

Perturbation Analysis

- Complex eigenvalues/instabilities arise from multiple root perturbation
- Resonance trace: $\omega(k,m) + n\omega(K,M) = \omega(k+nK, m+nM)$
  $\to$ where multiple roots live on.

Triad (n = 1) and Quartet (n = 2) Resonance

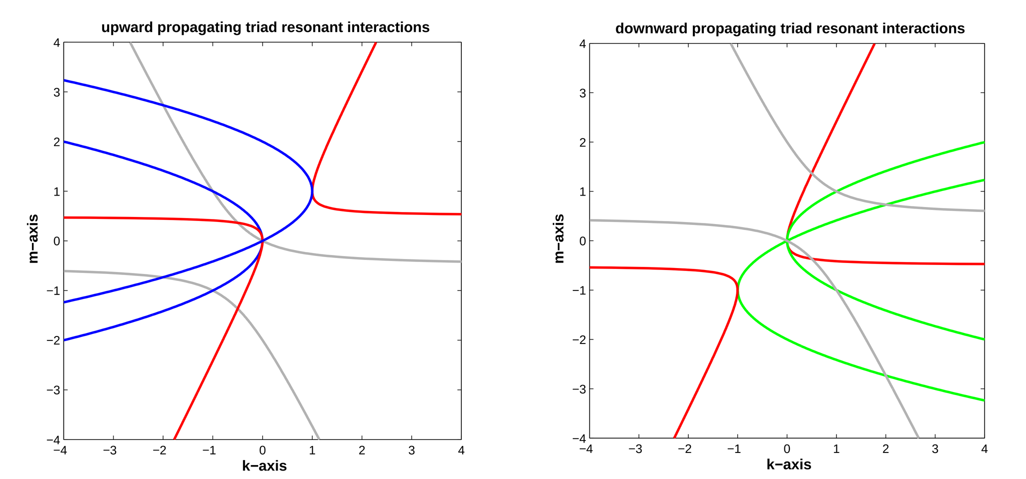

Figure 4: Triad resonant traces identified by corresponding resonance (color)

- Upward and downward $\vec c_g(k,m)$; Active and inert resonant traces
- ANSWER 1. By small ϵ pertubation, $\omega^{\pm}(k,m;\epsilon) \sim \pm\Omega(k,m)$

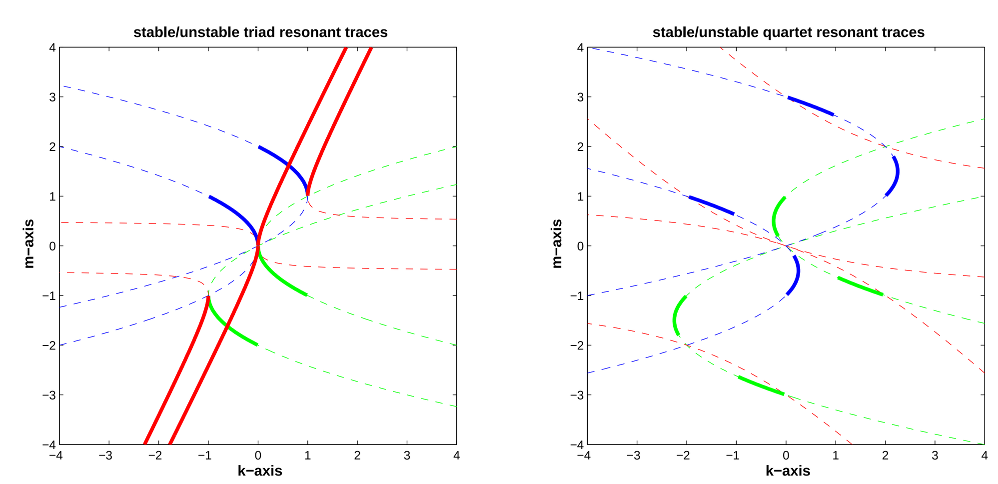

Figure 5: Unstable triad/quartet resonat traces via perturbation

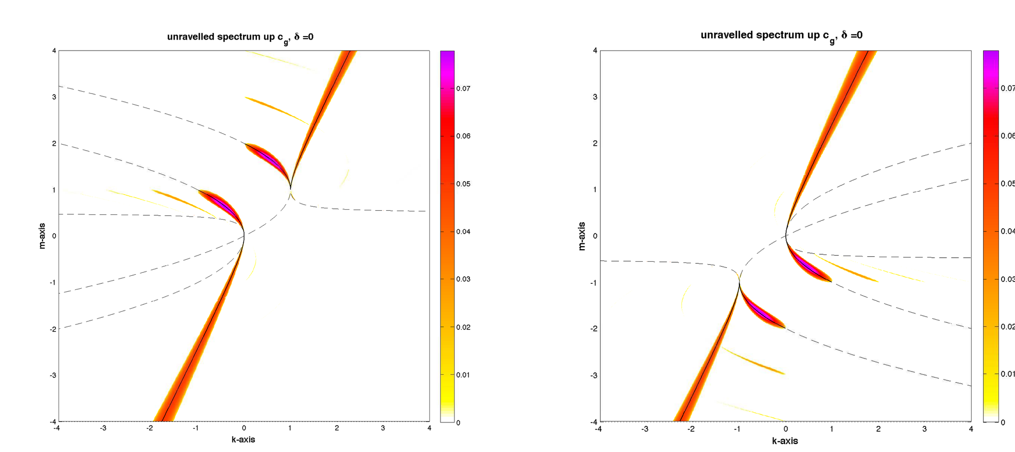

Figure 6: Spectrum Im $\omega^{\pm}$ vs frequency Re ω^{-}

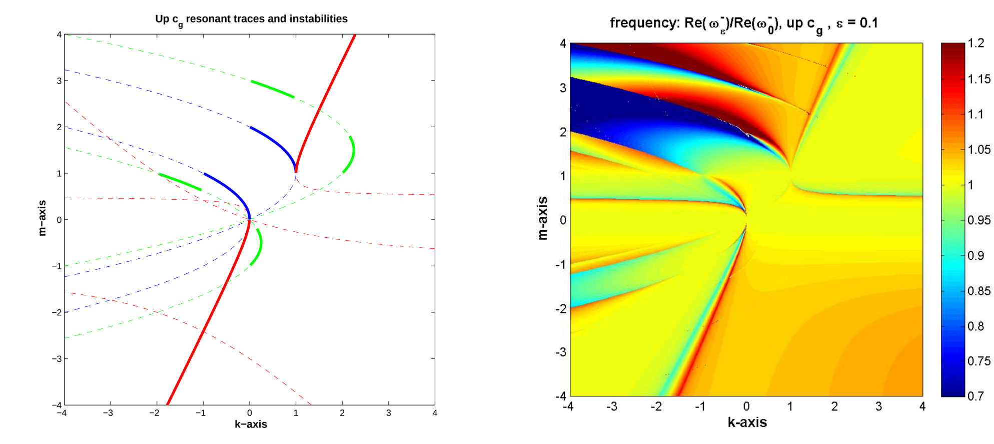

Figure 7: Spectrum Im $\omega^{\pm}$ vs frequency Re ω^{-}

Nonhydrostatic Limit Spectrum

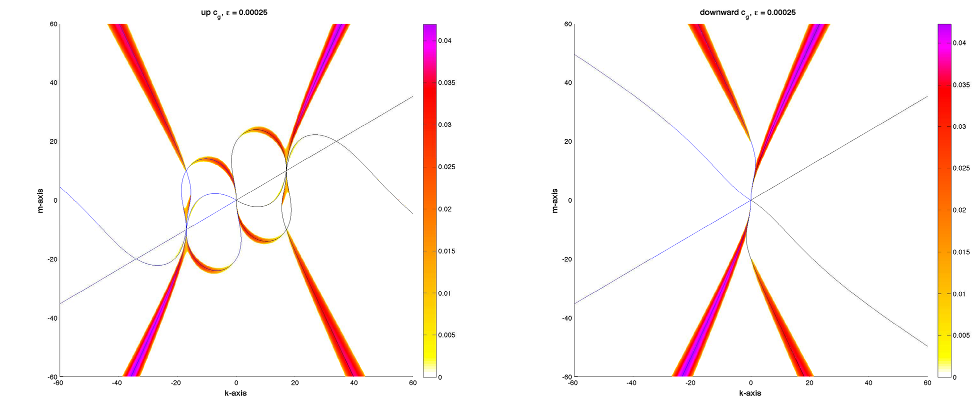

Figure 8: Spectrum vs frequency for the upward/downward $\vec c_g$ with $\delta = 1$

- Laplacian: $\delta = 1$; Primary wavenumbers: $(K,M) = (17{,}10)$.
- Spectrum of resonant instabilities (Im $(\omega(k,m))$) and frequency of resonant instabilities (Re $(\omega(k,m))$)
- Resonant traces correspond to jumps and branch-cuts in the Re $(\omega(k,m))$ figure

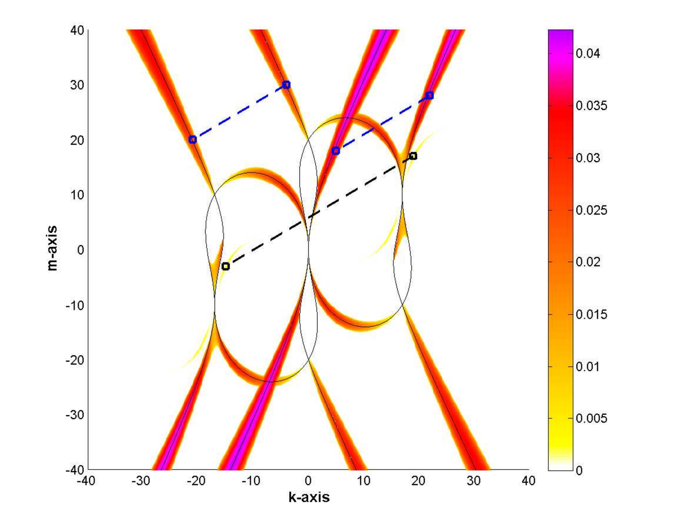

Figure 9: Correspondence between resonanting wave modes

Future Work

- Using Floquet spectral theory, to show in frequency plot (Re ω), resonant traces are continuous along instabilities and have branch-cuts along stabilities.
- To fully understand the wave resonance structure in the unravelled spectrum.

References

[1] D. J. Muraki, *Unravelling the Resonant Instabilities of a Wave in a Stratified Fluid*, 2007

[2] P. G. Drazin, *On the Instability of an Internal Gravity Wave*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. **356**, No. 1686 (1977), 411-432