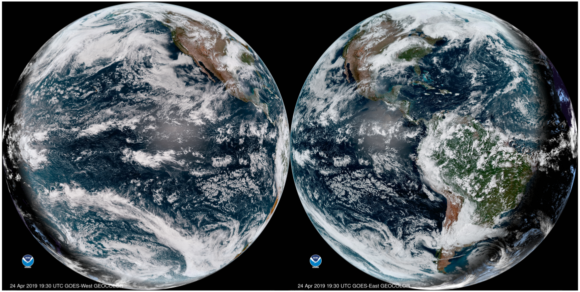
A Mathematical View of the Weather

- the basic equations that drive the weather patterns
- the calculus of fluid dynamics & thermodynamics
- making sense of real weather forecast maps



- David J Muraki (SFU)

NSERC
CRSNG

What is Mathematical Thinking?

Ideas, Thinking & Mathematical Science

- mathematics is a language for quantitative & deductive thought
- the universality of math as a language lies in the stories it can tell
- there is often a rift in perception of the *power of mathematical thought* ...
  - one view would say that math can be applied to understand anything
  - another would say that math can be applied to understand everything
- we often focus on teaching, "What is the math?"
  - yet more compelling might be, "Why we choose to reveal this particular math?"

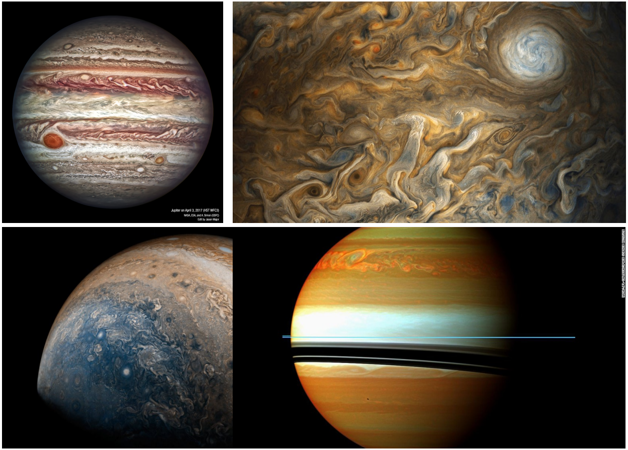
Models of the Weather

- a virtual laboratory for atmospheric science
  - simulation science: what will tomorrow's weather be like?
  - fundamental science: why is our weather the way it is?

What is Weather?

What determines the Earth's weather?

- ▷ zonal winds & vortices



Partial Differential Equations of Fluid Dynamics

A Description in Multi-Variable Calculus

- ▷ Euler's equations of inviscid fluid motion (1757) & Acheson (2000)
- ▷ fluid velocity, $\vec{u}(\vec{x},t) = \begin{pmatrix} u(\vec{x},t) \\ v(\vec{x},t) \\ w(\vec{x},t) \end{pmatrix}$; density, $\rho(\vec{x},t)$; pressure, $p(\vec{x},t)$
- ▷ gravity force, $\vec{F} = -\rho g\,\hat{z}$

quantité x comme variable. C'eſt pourquoi notre équation trouvée ſe réduit à celle-cy :

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z′, & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0.$$

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations ſuivantes :

$$P-\frac{1}{q}\left(\frac{dp}{dx}\right)=\left(\frac{du}{dt}\right)+u\left(\frac{du}{dx}\right)+v\left(\frac{du}{dy}\right)+w\left(\frac{du}{dz}\right)$$

$$Q-\frac{1}{q}\left(\frac{dp}{dy}\right)=\left(\frac{dv}{dt}\right)+u\left(\frac{dv}{dx}\right)+v\left(\frac{dv}{dy}\right)+w\left(\frac{dv}{dz}\right)$$

$$R-\frac{1}{q}\left(\frac{dp}{dz}\right)=\left(\frac{dw}{dt}\right)+u\left(\frac{dw}{dx}\right)+v\left(\frac{dw}{dy}\right)+w\left(\frac{dw}{dz}\right)$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho \boldsymbol{u})=0, \qquad (1.37)$$

$$\nabla\cdot\boldsymbol{u}=0,$$

as the basic equations of motion for an ideal fluid. They are known as *Euler's equations,* and written out in full they become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g,$$

What is the natural interpretation of these equations?

The Meaning of Euler I

Follow a Fluid Particle

- isolate a point "particle" moving with the flow: $\vec{X}(t)$
  - point moves with the flow velocity: $\dfrac{d\vec{X}}{dt} = \vec{u}(\vec{X}(t), t)$
- derivative of density at $\vec{X}(t)$
  - $\dfrac{d}{dt}\,\rho(\vec{X}(t), t) = \dfrac{\partial \rho}{\partial t} + \dfrac{dx}{dt}\,\dfrac{\partial \rho}{\partial x} + \dfrac{dy}{dt}\,\dfrac{\partial \rho}{\partial y} + \dfrac{dz}{dt}\,\dfrac{\partial \rho}{\partial z}\bigg|_{\vec{x}=\vec{X}(t)}$
  - $\qquad\qquad\qquad\qquad = \dfrac{D\rho}{Dt}\bigg|_{\vec{x}=\vec{X}(t)}$

Conservation of Mass

- change of local density from convergence/divergence
  - $\dfrac{d\rho}{dt}\bigg|_{\vec{X}(t)} = -\,(\nabla \cdot \vec{u})\,\rho\,\bigg|_{\vec{X}(t)}$
- since all fluid points lie on some $\vec{X}(t)$, then this is true for general $\vec{x}$

The Meaning of Euler II

Follow a Fluid Particle

- isolate a point "particle" moving with the flow: $\vec{X}(t)$

Conservation of Mass & Momentum

- change of local density from convergence/divergence
  - $\dfrac{D\rho}{Dt} = -\left(\nabla \cdot \vec{u}\right)\rho$
- change of local momentum is Newton's law
  - $\dfrac{D\rho\vec{u}}{Dt} = -\vec{\nabla}p - \rho g\,\hat{z}$
- why doesn't gravity keep accelerating "particles" downward?
  - nature establishes a hydrostatic pressure that (nearly) cancels: $p_z \approx -\rho g$
- four equations in five variables: $\vec{u}, \rho, p$
  - need some thermodynamics ...

The Speed of Sound

Waves in an Ideal Gas

- ideal gas law introduces temperature: $p = \rho R T$
- simplify Euler's: linearize $\left(\frac{D}{Dt} \to \frac{\partial}{\partial t}\right)$, no gravity & introduce $\log$
  - 2nd-order PDE that is trying to be the wave equation

$$\left.\begin{aligned} \frac{\partial(\log\rho)}{\partial t} &= -\left(\nabla\cdot\vec{u}\right) \\ \frac{\partial\vec{u}}{\partial t} &= -RT\ \vec{\nabla}(\log p) \end{aligned}\right\} \quad \frac{\partial^2(\log\rho)}{\partial t^2} - RT\ \nabla^2(\log p) \approx 0$$

Early Theories for the Speed of Sound in Air

- Newton formula (1687), if sound waves are isothermal (T constant) ...
  - log form of ideal gas: $\log p = \log\rho + \log RT$
  - $c = \sqrt{RT} \approx 279\ m/s \quad \to$ 15% too small at 0°C
- Laplace formula (1816), if sound waves are adiabatic (s constant) ...
- entropy of air: $\log s = C\log T - R\log p$
  - $c = \sqrt{\gamma RT} \approx 330\ m/s$

Incompressible Vorticity Dynamics

Incompressible Euler $(\nabla \cdot \vec{u}) = 0$

- Helmholtz (1858), vorticity vector: $\vec{\Omega} = \nabla \times \vec{u}$
  - curl of Euler equation (ρ constant):

$$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \vec{\nabla})\, \vec{u}$$

  - for 2D flow $(\partial/\partial z \to 0, w \equiv 0)$ gives a scalar vorticity variable: $\vec{\Omega} = \omega(x, y, t)\, \hat{z}$

$$\frac{D\omega}{Dt} = 0$$

Zero-Divergence 2D Velocity

- 2D streamfunction, $\psi(x, y, t)$, where $\vec{u} = \nabla \times (\psi\, \hat{z})$:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

- vorticity inversion (with BCs)

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \nabla^2 \psi$$

Waves in Flow

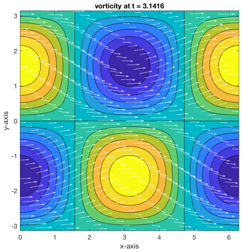
An Exact Travelling Wave Solution

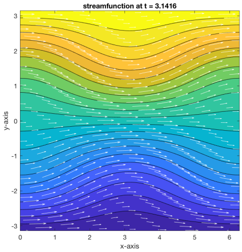
- streamfunction, $\psi(x, y, t)$:

$$\psi(x, y, t) = cy + A\cos(k(x - ct))\ \sin(ly)$$

- vorticity, $\omega(x, y, t)$:

$$\omega(x, y, t) = -(k^2 + l^2)\, A\cos(k(x - ct))\ \sin(ly)$$





- a natural coexistence of vortices in flow

Simple Equations for Atmospheric Weather

Euler Equations with Coriolis Rotation (1835)

- incompressible and adiabatic

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \frac{D\vec{u}}{Dt} + f(\hat{z} \times \vec{u}) &= -\frac{1}{\rho}\,\vec{\nabla} p - g\,\hat{z} \\ \frac{Ds}{Dt} &= 0 \end{aligned}$$

- hydrostatic pressure & Coriolis effect $\rightarrow$ thermal wind

An Extended Version of Vorticity

- Ertel potential vorticity (1942)

$$\frac{D}{Dt}\left(\frac{(f\,\hat{z} + \nabla \times \vec{u}) \cdot (\vec{\nabla} s)}{\rho}\right) = 0$$

- this makes it possible to analyze how vortex formation creates midlatitude weather

Simple Equations for Atmospheric Weather

Euler Equations with Coriolis Rotation (1835)

- incompressible and adiabatic

$$
\begin{aligned}
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\,\frac{\partial p}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\,\frac{\partial p}{\partial z} - g \\
\frac{Ds}{Dt} &= 0
\end{aligned}
$$

- hydrostatic pressure & Coriolis effect $\rightarrow$ thermal wind $\rightarrow$ jetstream

An Extended Version of Vorticity

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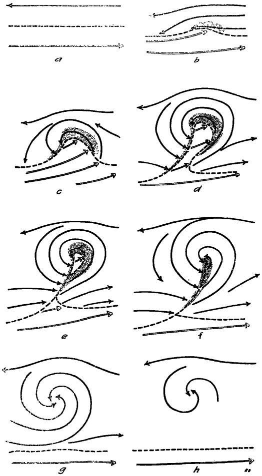
Life Cycle of Cyclones, Bjerknes & Solberg (1922)

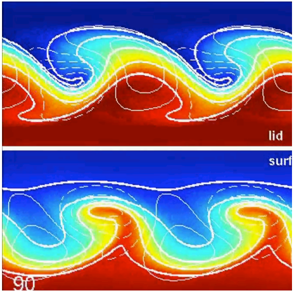


Waves in the Atmosphere

- Rossby (1939)
  - identifies waves in the midlatitude atmosphere
  - equator-to-pole temperature gradient & jetstream
- *On the Scale of Atmospheric Motions*, Charney (1948)
  - simple mathematical solutions for propagating Rossby waves (QG theory)
- Charney (1947) & Eady (1949)
  - baroclinic instability of Rossby waves in the tropospheric jetstream

Baroclinic Instability & the Formation of Storms

- asymmetric dynamics to surface low pressure
- leads to a "turbulent" weather pattern



An Unstable Atmosphere

Waves in the Atmosphere

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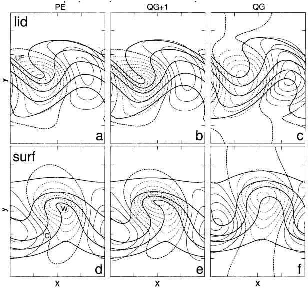
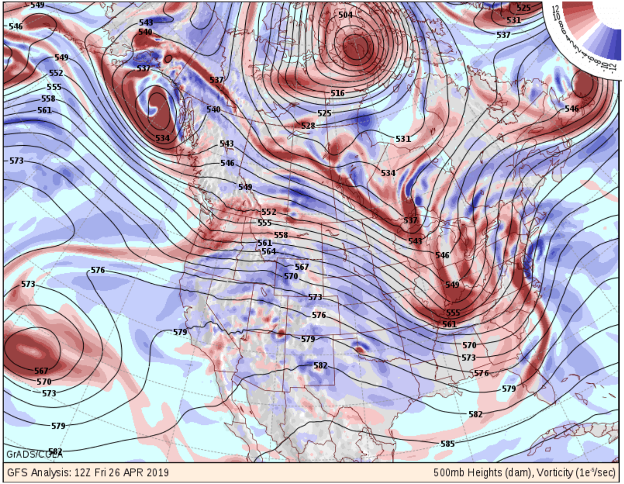
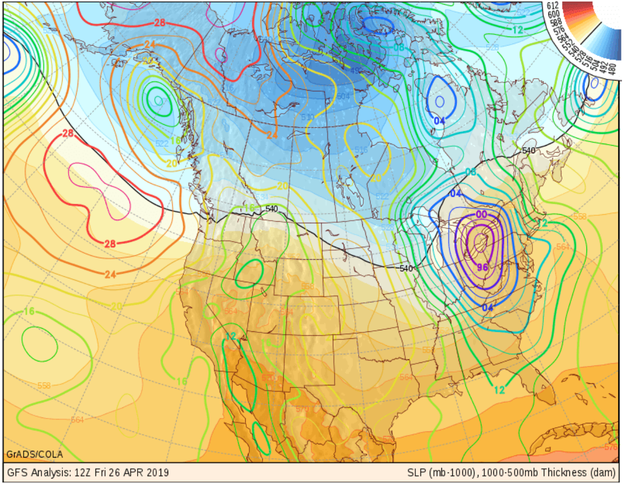


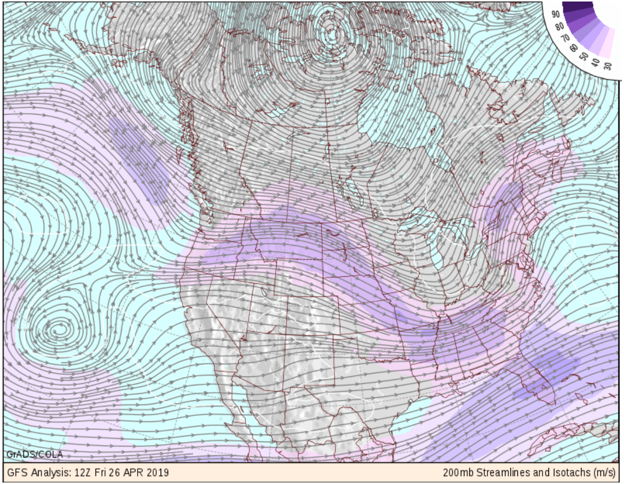
(12Z, 26 Apr 2019 – 5am PDT) 500 mB Vorticity & Height



(12Z, 26 Apr 2019 – 5am PDT) Sea-Level Pressure

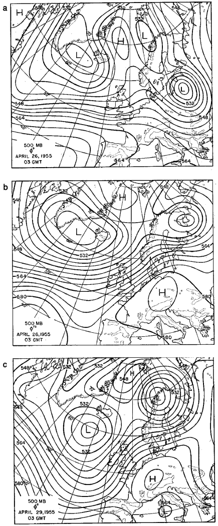


(12Z, 26 Apr 2019 – 5am PDT) 200 mB Streamline & Windspeed



Numerical Weather Prediction

- ▷ first forecast (∼ 1950), first operational forecasts (∼ 1955)



By late 1953, Rossby's Stockholm group was very busy on BESK, which Phillips reported was running quite well. Although output was slow (because of the printer), BESK was faster than the IAS machine on which it was modeled. When their magnetic drum arrived, the Stockholm team planned to increase grid size, the number of model layers, and forecast length.[72]

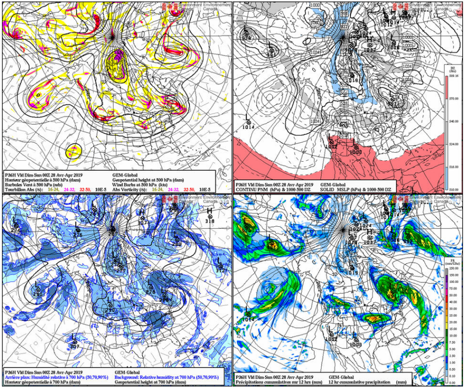
In early spring 1954, Smagorinsky went to Europe and reported that the British and the Swedes anticipated making daily operational predictions

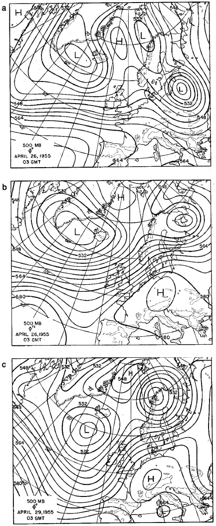
within 6 months.[73] It happened sooner than that. In mid June, Rossby informed Charney that the Stockholm team had made 23 barotropic forecasts for the eastern Atlantic and northern Europe, including two operational ones, on BESK. Having gotten good results, they were preparing to make operational 48-hour forecasts.[74] In contrast, the JNWPU's computer would not be available for at least another 6 months.

Computational Weather Forecasts

Numerical Weather Prediction

- ▷ first forecast ($\sim$ 1950), first operational forecasts ($\sim$ 1955)
- ▷ Environment Canada (36h forecast for 0Z, 28 Apr 2019)





Sensitivity to Initial Conditions

Non-periodic Deterministic Motions, Lorenz (1963)

- a simple set of ODEs extracted from the fluid equations

  - vorticity & entropy

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}, \tag{17}$$

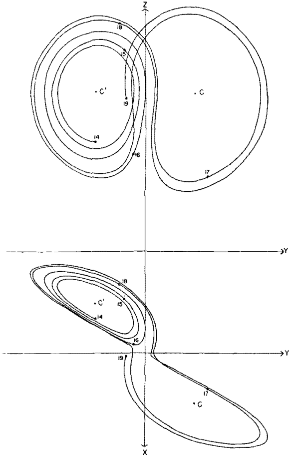
$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta. \tag{18}$$

  - mode equations

$$\dot{X} = -\sigma X + \sigma Y, \tag{25}$$

$$\dot{Y} = -XZ + rX - Y, \tag{26}$$

$$\dot{Z} = XY - bZ. \tag{27}$$



Dynamic Meteorology

- human-level understanding

Computational Forecasting

- large-scale computing

Statistical Methods

- large-data
- forecast sensitivity