PERTURBATION OF EIGENFUNCTIONS

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1. Notation

We first fix some notation. Let $\hat{\Omega}$ denote the reference triangle $(0,0),(1,0),(0,1)$. Let $\Omega_1 := (0,0),(1,0),(a_1,b_1)$ be an acute triangle, and $\Omega_2 := (0,0),(1,0),(a_2,b_2)$ be another acute triangle. Points in the reference triangle will have hats, and those in Ω_1 or Ω_2 will have the associated subscripts.

We denote by $F_1 : \hat{\Omega} \to \Omega_1$ and $F_2 := \hat{\Omega} \to \Omega_2$ affine mappings which take us from the reference triangle to the other acute triangles, and by $F_\epsilon : \Omega_1 \to \Omega_2$. Concretely, we can write these in terms of 2×2 invertible matrices B_1, B_2, B_ϵ as

$$F_1(\hat{\mathbf{x}}) = B_1\hat{x} = \mathbf{x}_1 \in \Omega_1, \qquad B_1 := \begin{pmatrix} 1 & a_1 \\ 0 & b_1 \end{pmatrix}$$

$$F_2(\hat{\mathbf{x}}) = B_2\hat{x} = \mathbf{x}_1 \in \Omega_2, \qquad B_2 := \begin{pmatrix} 1 & a_2 \\ 0 & b_2 \end{pmatrix}$$

$$F_\epsilon(\mathbf{x}_1) = B_\epsilon \mathbf{x}_1 = \mathbf{x}_2 \in \Omega_2,$$

We observe that $F_2 = F_\epsilon \circ F_1$, and so $B_2 = B_\epsilon B_1$.

We'll denote by $(\cdot,\cdot),(\cdot,\cdot)_1,(\cdot,\cdot)_2$ the L^2-inner products on $\hat{\Omega},\Omega_1,\Omega_2$. We'll also extensively use the following symmetric matrices:

$$M_1 := B_1^{-1}(B_1^{-1})^T = \frac{1}{b_1^2}\begin{pmatrix} b_1^2+a_1^2 & -a_1 \\ -a_1 & 1 \end{pmatrix}, \tag{1}$$

$$M_2 := B_2^{-1}(B_2^{-1})^T = \frac{1}{b_2^2}\begin{pmatrix} b_2^2+a_2^2 & -a_2 \\ -a_2 & 1 \end{pmatrix}, \tag{2}$$

The singular values of these symmetric matrices are:

$$\sigma_1(M_1) = \frac{1+b_1^2+a_1^2-\sqrt{1-2b_1^2+2a_1^2+b_1^4+2b_1^2a_1^2+a_1^4}}{b_1^2}, \tag{3}$$

$$\sigma_2(M_2) = \frac{1+b_1^2+a_1^2+\sqrt{1-2b_1^2+2a_1^2+b_1^4+2b_1^2a_1^2+a_1^4}}{b_1^2}, \tag{4}$$

We assume that there is a self-adjoint matrix $A = O(1)$ so that

$$M_\epsilon := B_\epsilon^{-1}(B_\epsilon^{-1})^T = I_{id} + \epsilon A. \tag{5}$$

Now

$M_2 = B_2^{-1}(B_2^{-1})^T = (B_\epsilon B_1)^{-1}((B_\epsilon B_1)^{-1})^T = B_1^{-1}B_\epsilon^{-1}(B_1^{-1}B_\epsilon^{-1})^T = B_1^{-1}B_\epsilon^{-1}(B_\epsilon^{-1})^T(B_1^{-1})^T$

and therefore

$$M_2 = B_1^{-1}M_\epsilon(B_1^{-1})^T = B_1^{-1}(B_1^{-1})^T + \epsilon B_1^{-1}A(B_1^{-1})^T = M_1 + \epsilon N, \tag{6}$$

where $N := B_1^{-1}A(B_1^{-1})^T$

Finally, we shall denote by $\sigma_1(A), \sigma_2(A)$ the smallest and largest singular values of a matrix A, respectively.

2. Eigenvalue problems

Consider the following two eigenvalue problems corresponding to the 2nd Neumann eigenvalue:

Eigenproblem1. *Find* $(w_1, \lambda_{1,2}) \in H^1(\Omega_1) \times \mathbb{R}^+$ *so that*

$$(\nabla w_1, \nabla v)_1 = \lambda_{1,2}(w_1, v)_1 \ \forall v \in H^1(\Omega_1), \qquad (w_1, 1)_1 = 0 \tag{7a}$$

We shall denote the next Neumann eigenvalue of this problem as $\lambda_{1,3} > \lambda_{1,2}$, ie, not counting multiplicity.

Eigenproblem2. *Find* $(w_2, \lambda_{2,2}) \in H^1(\Omega_2) \times \mathbb{R}^+$ *so that*

$$(\nabla w_2, \nabla v)_2 = \lambda_{2,2}(w_2, v)_2 \ \forall v \in H^1(\Omega_2), \qquad (w_2, 1)_2 = 0 \tag{7b}$$

(7)(a-b) can be reformulated on the same domain so we can compare the corresponding eigenfunctions. We choose to pull back to the reference domain $\hat{\Omega}$. Denoting $\hat{w}_1 := w_1 \circ F_1, \hat{w}_2 := w_2 \circ F_2$, we get the following equivalent problems:

ReferenceEigenproblem1. *Find* $(\hat{w}_1, \lambda_{1,2}) \in H^1(\hat{\Omega}) \times \mathbb{R}^+$ *so that*

$$(M_1 \nabla \hat{w}_1, \nabla v) = \lambda_{1,2}(\hat{w}_1, v) \ \forall v \in H^1(\hat{\Omega}), (\hat{w}_1, \hat{w}_1) = 1 \tag{8a}$$

$$(\hat{w}_1, 1) = 0 \tag{8b}$$

ReferenceEigenproblem2. *Find* $(\hat{w}_2, \lambda_{2,2}) \in H^1(\hat{\Omega}) \times \mathbb{R}^+$ *so that*

$$(M_2 \nabla \hat{w}_2, \nabla v) = \lambda_{2,2}(\hat{w}_2, v) \ \forall v \in H^1(\hat{\Omega}), \tag{9a}$$

$$(\hat{w}_2, 1) = 0 \tag{9b}$$

We wish to estimate $\nabla(\hat{w}_1 - \hat{w}_2)$ in various norms.

We first record some useful inequalities: Since $\hat{w}_2$ does not solve (8), we can decompose it as

$$\hat{w}_2 = \hat{w}_1 + \hat{v}, (\hat{v}, \hat{w}_1) = 0, \ \Rightarrow \|\hat{w}_2\|_0^2 = 1 + \|\hat{v}\|_0^2. \tag{10a}$$

That is, $\hat{v}$ must be nonzero and orthogonal to the eigenspace corresponding to (8). Indeed, for any nonzero vector v orthogonal to the eigenspace corresponding to (8), we actually have

$$(M_1 \nabla v, \nabla v) \geq \lambda_{1,3}\|v\|_0^2. \tag{10b}$$

From (8a) we also see, for any $\hat{v}$ which is orthogonal to $\hat{v}_1$, that

$$(M_1 \nabla \hat{w}_1, \nabla \hat{v}) = 0. \tag{10c}$$

Now, since $\hat{w}_2$ solves (9), it minimizes the Rayleigh quotient. In particular,

$$\frac{(M_2 \nabla \hat{w}_2, \nabla h w_2)}{(\hat{w}_2, \hat{w}_2)} = \frac{(M_2 \nabla \hat{w}_2, \nabla h w_2)}{1 + \|\hat{v}\|_0^2} \leq \frac{(M_2 \nabla \hat{w}_1, \nabla h w_1)}{(\hat{w}_1, \hat{w}_1)} = (M_2 \nabla \hat{w}_1, \nabla \hat{w}_1).$$

We get

$$(M_2 \nabla \hat{w}_2, \nabla \hat{w}_2) \leq (1 + \|\hat{v}\|_0^2)(M_2 \nabla \hat{w}_1, \nabla \hat{w}_1) \tag{11}$$

We expand $\hat{w}_2$ and use the inequalities above with (8) and (6):

$$\begin{aligned}
(M_2 \nabla \hat{w}_2, \nabla \hat{w}_2) &= (M_2 \nabla (\hat{w}_1 + \hat{v}), \nabla (\hat{w}_1 + \hat{v})) \\
&= (M_2 \nabla \hat{w}_1, \nabla \hat{w}_1) + (M_2 \nabla \hat{v}, \nabla \hat{v}) + 2(M_2 \nabla \hat{w}_1, \nabla \hat{v}) \\
&= (M_1 \nabla \hat{v}, \nabla \hat{v}) + \epsilon(N, \nabla \hat{v}, \nabla \hat{v}) + (M_2 \nabla \hat{w}_1, \nabla \hat{w}_1) + 2(M_1 \nabla \hat{w}_1, \nabla \hat{v}) + 2\epsilon(N \nabla \hat{w}_1, \nabla \hat{v})
\end{aligned}$$

and get

$$(M_2 \nabla \hat{w}_2, \nabla \hat{w}_2) = (M_1 \nabla \hat{v}, \nabla \hat{v}) + \epsilon(N, \nabla \hat{v}, \nabla \hat{v}) + (M_2 \nabla \hat{w}_1, \nabla \hat{w}_1) + 2\epsilon((N \nabla \hat{w}_1)^{\perp}, \nabla \hat{v}) \tag{12}$$

Here we have used the notation $(N \nabla \hat{w}_1) = (N \nabla \hat{w}_1)^{\parallel} + (N \nabla \hat{w}_1)^{\perp}$ where $(N \nabla \hat{w}_1)^{\parallel} \in span \nabla \hat{w}_1)$

Putting together (11) and (12), we get

$$\begin{aligned}
(M_1 \nabla \hat{v}, \nabla \hat{v}) + \epsilon(N \nabla \hat{v}, \nabla \hat{v}) + 2\epsilon((N \nabla \hat{w}_1)^{\perp}, \nabla \hat{v}) &\leq \|\hat{v}\|_0^2 (M_2 \nabla \hat{w}_1, \nabla \hat{w}_1) \\
&= \|\hat{v}\|_0^2 \left((M_1 \nabla \hat{w}_1, \nabla \hat{w}_1) + (N \nabla \hat{w}_1, \nabla \hat{w}_1)\right) \\
&= \|\hat{v}\|_0^2 \left(\lambda_{1,2} + \epsilon(N M_1^{-1} M_1 \nabla \hat{w}_1, \nabla \hat{w}_1)\right) \\
&\leq \|\hat{v}\|_0^2 \left(\lambda_{1,2} + \epsilon\sigma_2(N M_1^{-1})(M_1 \nabla \hat{w}_1, \nabla \hat{w}_1)\right) \\
&\leq \|\hat{v}\|_0^2 \left(\lambda_{1,2} + \epsilon\sigma_2(N M_1^{-1})\lambda_{1,2}\right) \\
&\leq \frac{\left(\lambda_{1,2} + \epsilon\sigma_2(N M_1^{-1})\lambda_{1,2}\right)}{\lambda_{1,3}} \|\nabla \hat{v}\|_0^2
\end{aligned}$$

We now have to estimate $\epsilon(N \nabla \hat{w}_1)^{\perp}, \nabla \hat{v})$. We first observe that

$$\begin{aligned}
((N \nabla \hat{w}_1)^{\perp}, \nabla \hat{v}) &= \left((N \nabla \hat{w}_1)^{\perp} - \frac{\sigma_1(N) + \sigma_2(N)}{2} \nabla \hat{w}_1, \nabla \hat{v}\right) + \left(\frac{\sigma_1(N) + \sigma_2(N)}{2} \nabla \hat{w}_1, \nabla \hat{v}\right) \\
&= \left((N \nabla \hat{w}_1)^{\perp} - \frac{\sigma_1(N) + \sigma_2(N)}{2} \nabla \hat{w}_1, \nabla \hat{v}\right) \\
&= \left(\left[N - \frac{\sigma_1(N) + \sigma_2(N)}{2}) \nabla \hat{w}_1\right]^{\perp}, \nabla \hat{v}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(N\nabla\hat{w}_1)^{\perp}\|_0 &= \| \left[N - \frac{\sigma_1(N)+\sigma_2(N)}{2})\nabla\hat{w}_1 \right]^{\perp} \|_0 \\
&\leq \sigma_2\left(N - \frac{\sigma_1(N)+\sigma_2(N)}{2}\right)\|\nabla\hat{w}_1\|_0 \\
&= \frac{\sigma_2(N)-\sigma_1(N)}{2}\left(M_1^{-1}M_1\nabla\hat{w}_1,\nabla\hat{w}_2\right)^{1/2} \\
&\leq \frac{\sigma_2(N)-\sigma_1(N)}{2}\sqrt{\sigma_2(M_1^{-1})}(M_1\nabla\hat{w}_1,\nabla\hat{w}_1)^{1/2} \\
&\leq \frac{\sigma_2(N)-\sigma_1(N)}{2}\sqrt{\sigma_2(M_1^{-1})}\lambda_{1,2}^{1/2}
\end{aligned}$$

We're almost done. We can estimate $(M_1\nabla\hat{v},\nabla\hat{v}) + \epsilon(N\nabla\hat{v},\nabla\hat{v}) \geq \sigma_1(M_1+\epsilon N)\|\nabla\hat{v}\|_0^2$, and therefore

$$\begin{aligned}
\sigma_1(M_1+\epsilon N)\|\nabla\hat{v}\|_0^2 &\leq (M_1\nabla\hat{v},\nabla\hat{v}) + \epsilon(N\nabla\hat{v},\nabla\hat{v}) \\
&\leq \frac{\left(\lambda_{1,2}+\epsilon\sigma_2(NM_1^{-1})\lambda_{1,2}\right)}{\lambda_{1,3}}\|\nabla\hat{v}\|_0^2 - 2\epsilon((N\nabla\hat{w}_1)^{\perp},\nabla\hat{v}) \\
&\leq \frac{\left(\lambda_{1,2}+\epsilon\sigma_2(NM_1^{-1})\lambda_{1,2}\right)}{\lambda_{1,3}}\|\nabla\hat{v}\|_0^2 + 2\epsilon\|(N\nabla\hat{w}_1)^{\perp}\|_0\|\nabla\hat{v}\|_0 \\
&\leq \frac{\left(\lambda_{1,2}+\epsilon\sigma_2(NM_1^{-1})\lambda_{1,2}\right)}{\lambda_{1,3}}\|\nabla\hat{v}\|_0^2 + 2\epsilon\left(\frac{\sigma_2(N)-\sigma_1(N)}{2}\sqrt{\sigma_2(M_1^{-1})}\lambda_{1,2}^{1/2}\right)\|\nabla\hat{v}\|_0
\end{aligned}$$

which leads us to

$$\begin{aligned}
\|\nabla\hat{v}\|_0 &\leq \left(\sigma_1(M_1+\epsilon N) - \frac{\left(\lambda_{1,2}+\epsilon\sigma_2(NM_1^{-1})\lambda_{1,2}\right)}{\lambda_{1,3}}\right)^{-1} 2\epsilon\left(\frac{\sigma_2(N)-\sigma_1(N)}{2}\sqrt{\sigma_2(M_1^{-1})}\lambda_{1,2}^{1/2}\right) \\
&= \left(\sigma_1(M_1)+\epsilon\sigma_1(N) - \frac{\lambda_{1,2}}{\lambda_{1,3}} - \epsilon\sigma_2(NM_1^{-1})\frac{\lambda_{1,2}}{\lambda_{1,3}}\right)^{-1} 2\epsilon\left(\frac{\sigma_2(N)-\sigma_1(N)}{2}\sqrt{\sigma_2(M_1^{-1})}\lambda_{1,2}^{1/2}\right)
\end{aligned}$$

We can compute these bounds explicitly in terms of the (exactly computable) singular values of the matrices involved.