OVERLAPPING SCHWARZ ITERATION

1. Introduction

Our goal is to either analytically solve, or numerically approximate, the 2nd Neumann eigenfunction of the Laplacian on a generic acute triangle $\Omega \equiv$ABC (see figure). Since we don't have an analytic solution on this domain, but can solve problems on sectors or the circle, we propose a domain decomposition strategy in the spirit of the overlapping Schwarz iteration.

Let 0 be the incenter of the triangle, and denote by Ω_0 the interior of the incircle. This is tangent to the segments AB, BC and CA at F, D, E respectively. Denote by Γ_{10} the segment of the circle connecting E, F. Denote by Ω_1 the sector AEF, and by Γ_{01} the arc connecting E, F. Note that Ω_1 and Ω_2 overlap. Repeat this process for the other three vertices. Let us denote the exact second Neumann eigenvalue as λ.

Now we'll proceed by iteration. At step n, suppose u_i^n for $i = 1, 2, 3$ satisfies

$$-\Delta u_i^n = \lambda_i^n u_i, \qquad x \in \Omega_i \tag{1}$$

$$\frac{\partial u_i^n}{\partial \nu} = 0 \quad x \in \partial\Omega_i \setminus \Gamma_{0i} \tag{2}$$

$$u_0^{n-1}\frac{\partial u_i^n}{\partial \nu} - \frac{\partial u_0^{n-1}}{\partial \nu} u_i^n = 0, \qquad x \in \Gamma_{0i} \tag{3}$$

(4)

The function u_0^n solves

$$-\Delta u_0^n = \lambda_0^n u_0^n, \quad x \in \Omega_0 \tag{5}$$

$$u_i^n \frac{\partial u_0^n}{\partial \nu} - \frac{\partial u_i^n}{\partial \nu} u_0^n = 0, \quad x \in \Gamma_{i0} \tag{6}$$

(7)

Now (u_i^n, λ_i) solve generalized eigenfunction problems on sectors of circles. If one knows $u_0^{n-1}, \frac{\partial u_0^{n-1}}{\partial \nu}$ on the curves Γ_{0i}, then one can use local Fourier-Bessel expansions to get u_i^n. Then one uses their traces onto Γ_{10}, and has the correct data to solve the eigenvalue problem for u_0^n on the disk. One can use Fourier-Bessel expansions to do this as well.

The claim is that as $n \to \infty$, the sequences u_i^n converge to the restriction of the actual eigenfunction u on the sub-domains. Clearly we have to prescribe a starting guess for the iteration.

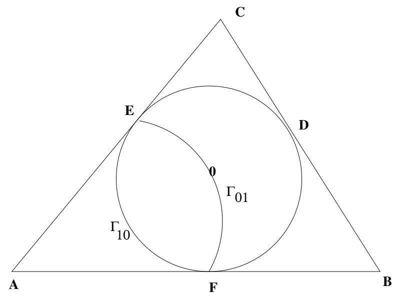

FIGURE 1. Domains

1.1. **Solving on the wedge Ω_i.** We are at step n of the iteration. Suppose $i = 1$ is fixed for concreteness, and the traces of u_0^{n-1} and $\frac{\partial u_0^{n-1}}{\partial \nu}$ on Γ_{01} are known.

$$-\Delta u_1^n = \lambda_1^n u_1, \qquad x \in \Omega_1 \tag{8}$$

$$\frac{\partial u_1^n}{\partial \nu} = 0 \quad x \in \partial\Omega_1 \setminus \Gamma_{01} \tag{9}$$

$$u_0^{n-1}\frac{\partial u_1^n}{\partial \nu} - \frac{\partial u_0^{n-1}}{\partial \nu} u_1^n = 0, \qquad x \in \Gamma_{01} \tag{10}$$

(11)

We will try the method of particular solutions of Fox and Henrici, adapted to this problem. We know that since the opening angle is α, in Ω_1 the functions

$$w_k(r, \theta) := J_{\frac{\pi k}{\alpha}} \sqrt{\lambda} r) \cos(\frac{\pi k}{\alpha}\theta) \tag{12}$$

will satisfy the Neumann conditions on the line segments AE, AF, as well as satisfy the equation $-\Delta w_k = \lambda w_k$.

So, we suppose $u_1^n = \sum_{k=1}^M c_k w_k(r, \theta)$. We want to find the coefficients c_k so as to satisfy the boundary condition on Γ_{01}. Now, Γ_{01} is an arc of radius $\rho_1 = AE$. Let (ρ_1, θ_j) be $2M$ collocation points along this curve. At each point, we want to enforce

$$0 = u_0^{n-1}(\rho_1, \theta_j)\frac{\partial u_1^n}{\partial \nu}(\rho_1, \theta_j) - \frac{\partial u_0^{n-1}}{\partial \nu}(\rho_1, \theta_j) u_1^n(\rho_1, \theta_j) \tag{13}$$

$$= u_0^{n-1}(\rho_1, \theta_j)\sum_{k=1}^M c_k \frac{\partial}{\partial r} J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j) - \frac{\partial u_0^{n-1}}{\partial \nu}(\rho_1, \theta_j)\sum_{k=1}^M c_k J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j) \tag{14}$$

This is equivalent to solving the rectangular nonlinear system

$$A(\lambda)\vec{c} = 0 \tag{15}$$

where $a_{jk}(\lambda) = u_0^{n-1}(\rho_1, \theta_j)\frac{\partial}{\partial r} J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j) - \frac{\partial u_0^{n-1}}{\partial \nu}(\rho_1, \theta_j) J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j)$

We find the solutions by looking for values of λ so that the smallest singular value of $A(\lambda)$ approaches 0. This is the Moler approach to the original Fox-Henrici-Moler paper.

Once we locate the solution $\vec{c}$, we have the iterate u_1^n. We do this same process for the other wedges as well.

1.2. **Solving on the disk Ω_0.** We are at step n of the iteration, and have solved for the functions u_i^n. We therefore have their traces on the arcs Γ_{i0}.

The function u_0^n solves

$$-\Delta u_0^n = \lambda_0^n u_0^n, \quad x \in \Omega_0 \tag{16}$$

$$u_i^n \frac{\partial u_0^n}{\partial \nu} - \frac{\partial u_i^n}{\partial \nu} u_0^n = 0, \quad x \in \Gamma_{i0} \tag{17}$$

(18)

We shall again use a Fourier-Bessel ansatz: let $z_m(r,\theta) = J_m(\lambda r)e^{im\theta)}$, and assume

$$u_0^n(r,\theta) = \sum_{k=0}^{M} d_m z_m(r,\theta)$$

. Repeat the process above of enforcing the (non-standard) boundary conditions at collocation points along Γ_{i0}.

2. Convergence?

At the end of the nth step, we have 4 functions: u_i^n, i=0,1,2,3 and 4 eigenvalues. We repeat the process until the eigenvalues are all the same number.