In these notes we compute L^2 and H^1 inner products of Fourier-Bessel basis functions in sectors, in the overlap between two sectors centered at vertices and going to the orthocenter.

We fix notation. First, a given acute triangle PQR (or $A_1A_2A_3$) has (as in previous notes) a side PQ of unit length, P has coordinates $(0,0)$ and Q has coordinates $(1,0)$. Now let PQR be fixed. Let the interior angles be A_i, $i = 1..3$. Denote by $\alpha_i := \frac{\pi}{\alpha_i}$. Let λ be a guess for the first non-zero Neumann eigenfunction on PQR.

Denote as

1. L^2 inner products

$$\phi_a(x,y,n) := \Gamma(\frac{n\pi}{A_1})J_{\frac{n\pi}{A_1}}(\sqrt{\lambda}r_1)\cos(n\frac{\pi}{A_1}\theta_1) \tag{1}$$

where $r_1 = \sqrt{x^2+y^2}, \theta_1 = atan(\frac{y}{x})$ are the polar coordinates of point $Z = (x,y)$ with the origin at point P = (0,0). Similarly, we define

$$\phi_b(x,y,n) := \Gamma(\frac{n\pi}{A_2})J_{\frac{n\pi}{A_2}}(\sqrt{\lambda}r_2)\cos(n\frac{\pi}{A_2}\theta_2) \tag{2}$$

where $r_2 = \sqrt{(1-x)^2+y^2}, \theta_1 = atan(\frac{y}{1-x})$ are the polar coordinates of point $Z = (x,y)$ with the origin at point Q = (1,0).

For illustration, we plot $\phi_a(x,y,n)$ and $\phi_b(x,y,m)$ in the same triangle, in sectors centered at P and Q respectively. We also plot the product $\phi_a(x,y,n)\phi_b(x,y,m)$. We finally tabulate the integral $\int_{overlap}\phi_a(x,y,n)\phi_b(x,y,m)\,dA$. The radii of the sectors are taken to be the distances of $|PO|$ and $|QO|$ respectively, where O is the orthocenter of the triangle (trilinear coordinates $\sec(A_1) : \sec(A_2) : \sec(A_3)$).

	n=0	n=1	n=2	n=3	n=4
m=0	0.063584163700319	0.000304951413692	-0.000005501884049	0.000000145447062	-0.000000004125635
m=1	0.000691996457238	0.000016146111819	0.000000056936884	-0.000000005879175	0.000000000228552
m=2	-0.000017269779724	0.000000401860666	0.000000008597667	-0.000000000128505	-0.000000000000457
m=3	0.000000526140782	-0.000000021474629	0.000000000280450	0.000000000010679	-0.000000000000456
m=4	-0.000000014262215	0.000000000781160	-0.000000000018997	0.000000000000003	0.000000000000025

Table 1. $\alpha = \frac{\pi}{5}, \beta = \frac{\pi}{3}$, $\int_{overlap}\phi_a(x,y,n)\phi_b(x,y,m)\,dA$.

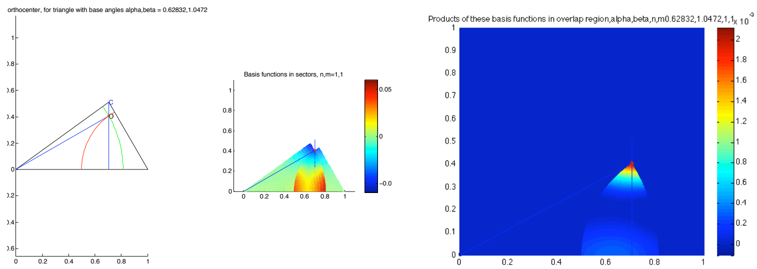

FIGURE 1. $\alpha = \frac{\pi}{5}, \beta = \frac{\pi}{3}$. L: 1-1 basis functions in sectors. R: product of Fourier-Bessel 1-1 in overlap region

	n=0	n=1	n=2	n=3	n=4
m=0	0.063584163700319	0.000691996457238	-0.000017269779724	0.000000526140782	-0.000000014262215
m=1	0.000304951413692	0.000016146111819	0.000000401860666	-0.000000021474629	0.000000000781160
m=2	-0.000005501884049	0.000000056936884	0.000000008597667	0.000000000280450	-0.000000000018997
m=3	0.000000145447062	-0.000000005879175	-0.000000000128505	0.000000000010679	0.000000000000003
m=4	-0.000000004125635	0.000000000228552	-0.000000000000457	-0.000000000000456	0.000000000000025

TABLE 2. $\alpha = \frac{\pi}{3}, \beta = \frac{\pi}{5}$, $\int_{overlap} \phi_a(x, y, n)\phi_b(x, y, m)\, dA$.

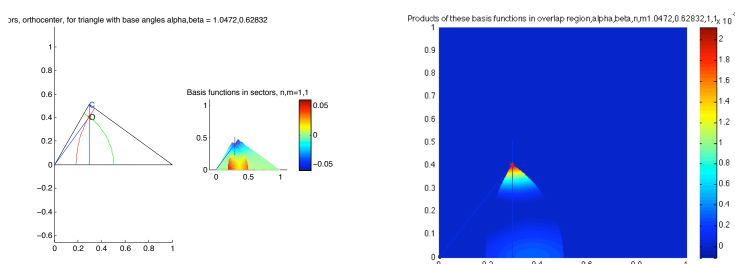

FIGURE 2. $\alpha = \frac{\pi}{3}, \beta = \frac{\pi}{5}$. L: 1-1 basis functions in sectors. R: product of Fourier-Bessel 1-1 in overlap region

	n=0	n=1	n=2	n=3	n=4
m=0	0.102928679164776	0.000236236958202	-0.000003147287321	0.000000090367936	-0.000000003660835
m=0	0.000236236958202	0.000038560464249	-0.000000457635117	0.000000007952567	-0.000000000287920
m=0	-0.000003147287321	-0.000000457635117	0.000000035314703	-0.000000001100485	0.000000000036239
m=0	0.000000090367936	0.000000007952567	-0.000000001100485	0.000000000060106	-0.000000000001723
m=0	-0.000000003660835	-0.000000000287920	0.000000000036239	-0.000000000001723	0.000000000000088

TABLE 3. $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{4}$, $\int_{overlap} \phi_a(x, y, n)\phi_b(x, y, m)\, dA$.

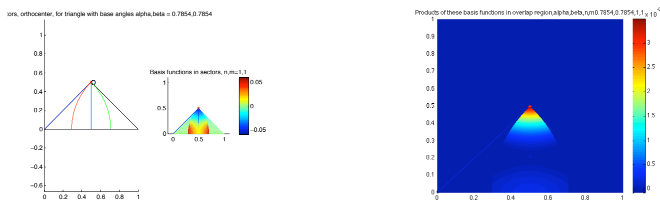

FIGURE 3. $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{4}$. L: 1-1 basis functions in sectors. R: product of Fourier-Bessel 1-1 in overlap region