A summary of approaches involving Bessel-Fourier expansion

July 13, 2013

1 Notation

We fix notation. First, a given acute triangle PQR (or $A_1A_2A_3$) has (as in previous notes) a side PQ of unit length, P has coordinates $(0,0)$ and Q has coordinates $(1,0)$. We examine the conjecture on a number of acute triangles by changing the coordinates of the third point, R.

Now let PQR be fixed. Let the interior angles be A_i, $i=1..3$. Denote by $\alpha_i := \frac{\pi}{\alpha_i}$. Let μ be a guess for the first non-zero Neumann eigenfunction on PQR.

Denote by $\phi_{ik}(r,\theta) := J_{k\alpha_i}(\sqrt{\mu}r)\cos(k\alpha_i\theta)$ a Bessel-Fourier basis function. Here $i = 1..3, k = 0..\infty$. Select points $z_j := (x_j, y_j)$ in PQR. Denote by r_{ij} the distance of z_j from vertex A_i, and by θ_{ij} the angle between segment A_iz_j and an edge containing A_i (in some consistent manner).

The next few sections summarize some approaches being considered using expansions involving ϕ_{ik}.

2 'Vanilla' method of particular solutions

- Write
$$u_N(z_j) := \sum_{i=1}^{3}\sum_{k=0}^{N-1} c_{ik}\phi_{ik}(r_{ij},\theta_{ij}) \tag{1}$$
.
- Pick $\{z_j\}_{j=1}^{3N}$ on the edges of PQR, and enforce $\nabla u_N(z_j)\cdot \mathbf{n}_j = 0$ at $j = 1..3N$. Here, $\mathbf{n}_j$ is the unit outward normal to the triangle PQR at point z_j.
- Pick $\{z_j\}_{3N+1}^{3N+M}$ randomly in the interior of PQR, and enforce $u_N(z_j) \neq 0$.

It doesn't matter what non-zero value is being assigned at the interior points.

Build the $(3N+M)\times 3N$ matrix C_μ associated with the steps above. The entries of C_μ depend nonlinearly on μ. For stability and conditioning reasons, compute the QR factorization of $Q_MR = C_\mu$. Throw away the bottom M rows of Q_M, call the resultant matrix Q. Find the minimal singular value $\sigma(\mu)$of Q. At the exact eigenvalue $\mu = \lambda$, $\sigma(\lambda) = 0$. Vary μ until $\sigma(\mu) = 0$. This gives the approximate eigenvalue, and the associated eigenvector.

3 Partition of unity

Let η_i be C^∞ functions which form a partition of unity of the triangle.

- Write
$$u_N(z_j) := \sum_{i=1}^{3} \sum_{k=0}^{N-1} c_{ik}\eta_i(r_{ij}\theta_{ij}\phi_{ik}(r_{ij}, \theta_{ij}) \tag{2}$$
.
- Pick $\{z_j\}_{j=1}^{3N}$ on the edges of PQR, and enforce $\nabla u_N(z_j) \cdot \mathbf{n}_j = 0$ at $j = 1..3N$. Here, $\mathbf{n}_j$ is the unit outward normal to the triangle PQR at point z_j.
- Pick $\{z_j\}_{3N+1}^{3N+M}$ randomly in the interior of PQR, and enforce $u_N(z_j) \neq 0$.

Again, it doesn't matter what non-zero value is being assigned at the interior points.

Build the $(3N + M) \times 3N$ matrix C_μ associated with the steps above. The entries of C_μ depend nonlinearly on μ. For stability and conditioning reasons, compute the QR factorization of $Q_M R = C_\mu$. Throw away the bottom M rows of Q_M, call the resultant matrix Q. Find the minimal singular value $\sigma(\mu)$of Q. At the exact eigenvalue $\mu = \lambda$, $\sigma(\lambda) = 0$. Vary μ until $\sigma(\mu) = 0$. This gives the approximate eigenvalue, and the associated eigenvector. Note now that $\eta_i(r, theta)\phi_{ik}(r, \theta)$ does NOT satisfy the eigenvalue equation $\Delta u = \lambda u$.

4 Overlapping Schwartz iteration

[Taken from previous notes]

Our goal is to either analytically solve, or numerically approximate, the 2nd Neumann eigenfunction of the Laplacian on a generic acute triangle $\Omega \equiv$ABC (see figure). Since we don't have an analytic solution on this domain, but can solve problems on sectors or the circle, we propose a domain decomposition strategy in the spirit of the overlapping Schwarz iteration.

Let 0 be the incenter of the triangle, and denote by Ω_0 the interior of the incircle. This is tangent to the segments AB, BC and CA at F, D, E respectively. Denote by Γ_{10} the segment of the circle connecting E, F. Denote by Ω_1 the sector AEF, and by Γ_{01} the arc connecting E, F. Note that Ω_1 and Ω_2 overlap. Repeat this process for the other three vertices. Let us denote the exact second Neumann eigenvalue as λ.

Now we'll proceed by iteration. At step n, suppose u_i^n for $i = 1, 2, 3$ satisfies

$$-\Delta u_i^n = \lambda_i^n u_i, \qquad x \in \Omega_i \tag{3}$$

$$\frac{\partial u_i^n}{\partial \nu} = 0 \quad x \in \partial\Omega_i \setminus \Gamma_{0i} \tag{4}$$

$$u_0^{n-1}\frac{\partial u_i^n}{\partial \nu} - \frac{\partial u_0^{n-1}}{\partial \nu} u_i^n = 0, \qquad x \in \Gamma_{0i} \tag{5}$$

$$\tag{6}$$

The function u_0^n solves

$$-\Delta u_0^n = \lambda_0^n u_0^n, \quad x \in \Omega_0 \tag{7}$$

$$u_i^n \frac{\partial u_0^n}{\partial \nu} - \frac{\partial u_i^n}{\partial \nu} u_0^n = 0, \quad x \in \Gamma_{i0} \tag{8}$$

$$\tag{9}$$

Now $(u_i^n,\ \lambda_i)$ solve generalized eigenfunction problems on sectors of circles. If one knows $u_0^{n-1}, \frac{\partial u_0^{n-1}}{\partial \nu}$ on the curves Γ_{0i}, then one can use local Fourier-Bessel expansions to get u_i^n. Then one uses their traces onto Γ_{10}, and has the correct data to solve the eigenvalue problem for u_0^n on the disk. One can use Fourier-Bessel expansions to do this as well.

The claim is that as $n \to \infty$, the sequences u_i^n converge to the restriction of the actual eigenfunction u on the sub-domains. Clearly we have to prescribe a starting guess for the iteration.

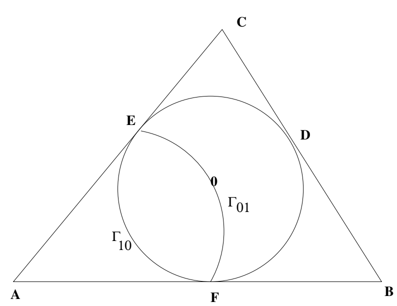

Figure 1: Domains

4.1 Solving on the wedge Ω_i

We are at step n of the iteration. Suppose $i = 1$ is fixed for concreteness, and the traces of u_0^{n-1} and $\frac{\partial u_0^{n-1}}{\partial \nu}$ on Γ_{01} are known.

$$-\Delta u_1^n = \lambda_1^n u_1, \qquad x \in \Omega_1 \tag{10}$$

$$\frac{\partial u_1^n}{\partial \nu} = 0 \quad x \in \partial\Omega_1 \setminus \Gamma_{01} \tag{11}$$

$$u_0^{n-1}\frac{\partial u_1^n}{\partial \nu} - \frac{\partial u_0^{n-1}}{\partial \nu}u_1^n = 0, \qquad x \in \Gamma_{01} \tag{12}$$

$$\tag{13}$$

We will try the method of particular solutions of Fox and Henrici, adapted to this problem. We know that since the opening angle is α, in Ω_1 the functions

$$\phi_{ik}(r,\theta) \tag{14}$$

with $\mu = \lambda$ will satisfy the Neumann conditions on the line segments AE, AF, as well as satisfy the equation $-\Delta w_k = \lambda w_k$.

So, we suppose $u_1^n = \sum_{k=1}^M c_k \phi_{ik}(r,\theta)$. We want to find the coefficients c_k so as to satisfy the boundary condition on Γ_{01}. Now, Γ_{01} is an arc of radius $\rho_1 = AE$. Let (ρ_1, θ_j) be $2M$ collocation points along this curve. At each point, we want to enforce

$$0 = u_0^{n-1}(\rho_1,\theta_j)\frac{\partial u_1^n}{\partial \nu}(\rho_1,\theta_j) - \frac{\partial u_0^{n-1}}{\partial \nu}(\rho_1,\theta_j)u_1^n(\rho_1,\theta_j) \tag{15}$$

$$= u_0^{n-1}(\rho_1,\theta_j)\sum_{k=1}^M c_k \frac{\partial}{\partial r}J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j) - \frac{\partial u_0^{n-1}}{\partial \nu}(\rho_1,\theta_j)\sum_{k=1}^M c_k J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j) \tag{16}$$

This is equivalent to solving the rectangular nonlinear system

$$A(\lambda)\vec{c} = 0 \tag{17}$$

where $a_{jk}(\lambda) = u_0^{n-1}(\rho_1,\theta_j)\frac{\partial}{\partial r}J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j) - \frac{\partial u_0^{n-1}}{\partial \nu}(\rho_1,\theta_j)J_{\frac{\pi k}{\alpha}}(\sqrt{\lambda}\rho_1)\cos(\frac{\pi k}{\alpha}\theta_j)$

We find the solutions by looking for values of λ so that the smallest singular value of $A(\lambda)$ approaches 0. This is the Moler approach to the original Fox-Henrici-Moler paper.

Once we locate the solution $\vec{c}$, we have the iterate u_1^n. We do this same process for the other wedges as well.

4.2 Solving on the disk Ω_0

We are at step n of the iteration, and have solved for the functions u_i^n. We therefore have their traces on the arcs Γ_{i0}.

The function u_0^n solves

$$-\Delta u_0^n = \lambda_0^n u_0^n, \quad x \in \Omega_0 \tag{18}$$

$$u_i^n \frac{\partial u_0^n}{\partial \nu} - \frac{\partial u_i^n}{\partial \nu}u_0^n = 0, \quad x \in \Gamma_{i0} \tag{19}$$

$$\tag{20}$$

We shall again use a Fourier-Bessel ansatz: let $z_m(r,\theta) = J_m(\lambda r)e^{im\theta)}$, and assume

$$u_0^n(r,\theta) = \sum_{k=0}^{M} d_m z_m(r,\theta)$$

. Repeat the process above of enforcing the (non-standard) boundary conditions at collocation points along Γ_{i0}.

4.3 Convergence?

At the end of the nth step, we have 4 functions: u_i^n, i=0,1,2,3 and 4 eigenvalues. We repeat the process until the eigenvalues are all the same number.

There is currently no proof that this method is convergent in a reasonable number of steps

5 Lior's idea

Write

$$u_N(z_j) := \sum_{k=0}^{N-1} c_{1k}\phi_{1k}(r_{1j},\theta_{1j}) = \sum_{k=0}^{N-1} c_{2k}\phi_{1k}(r_{2j},\theta_{ij}) \tag{21}$$

for points z_j in the intersection of the sectors centered at A_1 and A_2. Fix $c_{1,0} = 0$, pick enough points z_j, and solve for the unknown coefficients as described in `http://www.math.ubc.ca/ lior/work`

In this instance, the expansion for u_N is not the same as being used in the other approaches.