Math 252
Mid-Term Exam Solutions
17 Feb 1999

1. $(\vec{\nabla}\times(\vec{F}\times\vec{G}))_i = \varepsilon_{ijk}\,\partial_j(\vec{F}\times\vec{G})_k = \varepsilon_{ijk}\,\partial_j(\varepsilon_{klm}F_\ell G_m)$

$= \varepsilon_{ijk}\varepsilon_{klm}\,\partial_j(F_\ell G_m) = \varepsilon_{ijk}\varepsilon_{lmk}\,\partial_j(F_\ell G_m)$

$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\,\partial_j(F_\ell G_m)$

$= \partial_j(F_i G_j) - \partial_j(F_j G_i)$

$= G_j\partial_j F_i + F_i\partial_j G_j - G_i\partial_j F_j - F_j\partial_j G_i$

$= (\vec{G}\cdot\vec{\nabla})F_i + F_i\,\vec{\nabla}\cdot\vec{G} - G_i\,\vec{\nabla}\cdot\vec{F} - (\vec{F}\cdot\vec{\nabla})G_i$

$\therefore\ \vec{\nabla}\times(\vec{F}\times\vec{G}) = (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F}(\vec{\nabla}\cdot\vec{G}) - \vec{G}(\vec{\nabla}\cdot\vec{F})$

QED

2. There is a vector identity which says

$\vec{\nabla}\times\vec{\nabla}\phi = 0$ for any scalar field ϕ.

3.

$$\vec{\nabla}\vec{F} = (\hat{i}\partial_1 + \hat{j}\partial_2 + \hat{k}\partial_3)(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$\boxed{\begin{aligned} &= \partial_1 F_1 \hat{i}\hat{i} + \partial_1 F_2 \hat{i}\hat{j} + \partial_1 F_3 \hat{i}\hat{k} \\ &+ \partial_2 F_1 \hat{j}\hat{i} + \partial_2 F_2 \hat{j}\hat{j} + \partial_2 F_3 \hat{j}\hat{k} \\ &+ \partial_3 F_1 \hat{k}\hat{i} + \partial_3 F_2 \hat{k}\hat{j} + \partial_3 F_3 \hat{k}\hat{k} \end{aligned}}$$

4. Laplace's equation says $\vec{\nabla}^2 f = 0$

$$f = \sin(px)\sinh(qy)e^{rz}$$

$$\frac{\partial^2 f}{\partial x^2} = -p^2 f$$

$$\frac{\partial^2 f}{\partial y^2} = q^2 f$$

$$\frac{\partial^2 f}{\partial z^2} = r^2 f$$

$$\vec{\nabla}^2 f = (-p^2 + q^2 + r^2) f = 0$$

$$\therefore \boxed{p^2 = q^2 + r^2}$$

5. $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

(a) $\vec{\nabla}\cdot\vec{F} = 0$

(b) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = \vec{0}$$

(c)

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

Integrating $\int x\,dx = \int y\,dy \quad \therefore x^2 = y^2 + C_1$

$\int x\,dx = \int z\,dz \quad \therefore x^2 = z^2 + C_2$

At (x_0, y_0, z_0): $x_0^2 = y_0^2 + C_1 \quad \therefore C_1 = x_0^2 - y_0^2$

$x_0^2 = z_0^2 + C_2 \quad \therefore C_2 = x_0^2 - z_0^2$

$$\therefore \left.\begin{aligned} x^2 &= y^2 + (x_0^2 - y_0^2) \\ x^2 &= z^2 + (x_0^2 - z_0^2) \end{aligned}\right\} \quad \boxed{\begin{aligned} y^2 &= x^2 + (y_0^2 - x_0^2) \\ z^2 &= x^2 + (z_0^2 - x_0^2) \end{aligned}}$$

In parametric form:

$$\boxed{\begin{aligned} x &= t \\ y &= \pm\sqrt{t^2 + (y_0^2 - x_0^2)} \\ z &= \pm\sqrt{t^2 + (z_0^2 - x_0^2)} \end{aligned}}$$

6.

The surface can be described by $f=0$ where f is a scalar field given by

$$f = z - x^2 - y^2$$

$f=0$ is therefore an isotimic surface, and the gradient will be perpendicular to this surface:

$$\vec{\nabla} f = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$\vec{\nabla} f(3,4,25) = -6\hat{i} - 8\hat{j} + \hat{k} = \vec{n}$$

where $\vec{n}$ is normal to the surface at $(3,4,25)$.
The equation of the tangent plane is:

$$\vec{n}\cdot(\vec{R}-\vec{R}_0) = 0$$

$$\therefore \boxed{-6(x-3) - 8(y-4) + (z-25) = 0}$$

7.

$$\vec{R} = e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j}$$

$$\frac{d\vec{R}}{dt} = e^t(\cos t - \sin t)\hat{i} + e^t(\cos t + \sin t)\hat{j}$$

$$\frac{d^2\vec{R}}{dt^2} = e^t(\cancel{\cos t} - \sin t - \sin t - \cancel{\cos t})\hat{i}$$

$$+ e^t(\cos t + \cancel{\sin t} - \cancel{\sin t} + \cos t)\hat{j}$$

$$= -2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j}$$

(a)

speed $v = \left|\frac{d\vec{R}}{dt}\right| = e^t\sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}$

$= e^t\sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}\,e^t$

(b)

$a_t = \frac{dv}{dt} = \sqrt{2}\,e^t$

(c)

$a_n = \sqrt{a^2 - a_t^2}$

and $a^2 = \left|\frac{d^2\vec{R}}{dt^2}\right|^2 = 4e^{2t}(\sin^2 t + \cos^2 t) = 4e^{2t}$

$\therefore a_n = \sqrt{4e^{2t} - 2e^{2t}} = \sqrt{2e^{2t}} = \sqrt{2}\,e^t$

(d)

$\vec{T} = \frac{1}{v}\frac{d\vec{R}}{dt} = \frac{1}{\sqrt{2}}\left[(\cos t - \sin t)\hat{i} + (\cos t + \sin t)\hat{j}\right]$

(e) $k = \frac{a_n}{v^2} = \frac{\sqrt{2}\,e^t}{2e^{2t}} = \frac{1}{\sqrt{2}}e^{-t}$

(f)

The curve never leaves the xy-plane, so the torsion is 0.