Maxwell's equations

Notes on HW 7, extra problem 1: electric and magnetic fields and waves
(some discussion of how to approach parts (e) and (f), with some answers)

We have $$\vec{E} \underset{(\text{def})}{\equiv} \underbrace{[A\cos(\omega t - \vec{k}\cdot\vec{R}) + B\sin(\omega t - \vec{k}\cdot\vec{R})]}_{\equiv E}\,\hat{u}_E \equiv E\,\hat{u}_E,$$

where $E = E(\vec{R}, t) = \pm|\vec{E}|$

depending on the sign of $A\cos(\ ) + B\sin(\ )$

and similarly, $$\vec{B} = [C\cos(\omega t - \vec{k}\cdot\vec{R}) + D\sin(\omega t - \vec{k}\cdot\vec{R})]\,\hat{u}_B \equiv B\,\hat{u}_B.$$

Let $\vec{k} = k_1\hat{\imath} + k_2\hat{\jmath} + k_3\hat{k}$, $\vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

It is also convenient to define

$$H_E = A\sin(\omega t - \vec{k}\cdot\vec{R}) - B\cos(\omega t - \vec{k}\cdot\vec{R})$$
$$H_B = C\sin(\omega t - \vec{k}\cdot\vec{R}) - D\cos(\omega t - \vec{k}\cdot\vec{R}).$$

The motivation for these definitions: for instance

$$\frac{\partial\vec{E}}{\partial x} = \frac{\partial E}{\partial x}\hat{u}_E = \frac{\partial}{\partial x}\left[A\cos(\omega t - \vec{k}\cdot\vec{R}) + B\sin(\omega t - \vec{k}\cdot\vec{R})\right]\hat{u}_E$$
$$= k_1 H_E\,\hat{u}_E, \qquad \frac{\partial E}{\partial y} = k_2 H_E, \quad \frac{\partial E}{\partial z} = k_3 H_E, \ldots$$

since $$\frac{\partial}{\partial x}\cos(\omega t - \vec{k}\cdot\vec{R}) = -\sin(\omega t - \vec{k}\cdot\vec{R})\frac{\partial}{\partial x}(\omega t - (k_1x + k_2y + k_3z))$$
$$= k_1\sin(\omega t - \vec{k}\cdot\vec{R}),$$

and $$\frac{\partial}{\partial x}\sin(\omega t - \vec{k}\cdot\vec{R}) = -k_1\cos(\omega t - \vec{k}\cdot\vec{R}).$$

Now with $\hat{u}_E = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$, we have ($u_i$ constant, $(u_1^2 + u_2^2 + u_3^2 = 1)$)

$$\vec{E} = (Eu_1)\hat{\imath} + (Eu_2)\hat{\jmath} + (Eu_3)\hat{k}$$
$$\nabla\cdot\vec{E} = \frac{\partial}{\partial x}(Eu_1) + \frac{\partial}{\partial y}(Eu_2) + \frac{\partial}{\partial z}(Eu_3)$$
$$= \frac{\partial E}{\partial x}u_1 + \frac{\partial E}{\partial y}u_2 + \frac{\partial E}{\partial z}u_3 = k_1 H_E u_1 + k_2 H_E u_2 + k_3 H_E u_3$$

$$\Rightarrow \boxed{\nabla\cdot\vec{E} = H_E\,\vec{k}\cdot\hat{u}_E}\ ; \text{ similarly } \nabla\cdot\vec{B} = H_B\,\vec{k}\cdot\hat{u}_B$$

$$(\nabla \times \vec{E})_1 = \frac{\partial}{\partial y}(E u_3) - \frac{\partial}{\partial z}(E u_2) = k_2 H_E u_3 - k_3 H_E u$$

$$= H_E (k_2 u_3 - k_3 u_2) = H_E (\vec{k} \times \hat{u}_E),$$

and similarly for the y, z components

$$\Rightarrow \boxed{\nabla \times \vec{E} = H_E \vec{k} \times \hat{u}_E} \quad \text{and} \quad \nabla \times \vec{B} = H_B \vec{k} \times \hat{u}_B .$$

We also have

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial E}{\partial t} \hat{u}_E = \left[-\omega A \sin(\omega t - \vec{k}\cdot\vec{R}) + \omega B \cos(\omega t - \vec{k}\cdot\vec{R}) \right] \hat{u}_E$$

$$\Rightarrow \boxed{\frac{\partial \vec{E}}{\partial t} = -\omega H_E \hat{u}_E} \quad \text{and} \quad \frac{\partial \vec{B}}{\partial t} = -\omega H_B \hat{u}_B .$$

Thus we find, by substituting into $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, that

$$H_E \vec{k} \times \hat{u}_E = \omega H_B \hat{u}_B$$

- let $\vec{k} = |\vec{k}| \hat{\alpha}$, where $\hat{\alpha} = \vec{k}/|\vec{k}|$ is the unit vector in the direction of $\vec{k}$ (we cannot use $\hat{k}$..)

$$\Rightarrow H_E |\vec{k}| \hat{\alpha} \times \hat{u}_E = \omega H_B \hat{u}_B$$

& this shows that $\hat{u}_B \perp \hat{u}_E$ and $\hat{u}_B \perp \hat{\alpha}$

Since $\hat{\alpha}$, $\hat{u}_E$ and $\hat{u}_B$ are mutually perpendicular vectors, the vector part of the equation just gives the direction, and we can equate the coefficients.

For convenience, let $\hat{\alpha}$, $\hat{u}_E$, $\hat{u}_B$ form a right-handed system (this just affects the sign of C and D), so $\hat{\alpha} \times \hat{u}_E = \hat{u}_B$.

Then we have

$$H_E |\vec{k}| = \omega H_B \Rightarrow H_E = \omega/|\vec{k}| \, H_B = c H_B .$$

Substituting the definitions of H_E and H_B, we find (compare coefficients of $\sin(\,)$, $\cos(\,)$)

$$A = cC, \quad B = cD.$$

Summary picture: