Solution to Section 5.8 # 8, using tensor notation (this problem is quite long, and much worse if you write out the terms...)

The old basis is $\{\hat{\imath}, \hat{\jmath}, \hat{k}\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, and the new basis is written as $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\} = \{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$.

The coordinates $(x, y, z) = (x_1, x_2, x_3)$ wrt. the old basis and $(x', y', z') = (x_1', x_2', x_3')$ wrt the new basis are related by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = J \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \text{ie} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = J \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

where the transformation matrix J is orthogonal, $JJ^T = I$, and has components $J_{ij} = \hat{e}_i \cdot \hat{e}_j' = \dfrac{\partial x_i}{\partial x_j'}$; note that $\dfrac{\partial x_j'}{\partial x_i} = J_{ji} = (J^T)_{ij}$

Using the summation convention, the relation between the old and new coordinates is $x_i = J_{ij} x_j'$, and the chain rule implies that derivatives wrt. the new coordinates are related to derivatives wrt. the old coordinates via

$$\frac{\partial f}{\partial x_j'} = \frac{\partial f}{\partial x_k} \frac{\partial x_k}{\partial x_j'} \qquad \text{(summation convention)}$$

$$= J_{kj} \frac{\partial f}{\partial x_k} \qquad (J^T)_{kj}$$

The orthogonality relation for J is $(JJ^T)_{ij} = \delta_{ij} \Rightarrow J_{ik} J_{jk} = \delta_{ij}$

A vector field $\vec{F}$ can be written wrt. the old and new bases as

$$\vec{F} = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$$
$$= F_1'(x',y',z')\,\hat{\imath}' + F_2'(x',y',z')\,\hat{\jmath}' + F_3'(x',y',z')\,\hat{k}'$$

or more succinctly, using summation notation,

$$\vec{F} = F_i \hat{e}_i = F_j' \hat{e}_j'.$$

The components wrt the new basis are found from the old components as $F_j' = \vec{F} \cdot \hat{e}_j' = F_i \hat{e}_i \cdot \hat{e}_j' = J_{ij} F_i$, $F_i = J_{ij} F_j'$

or in matrix form,

$$\begin{pmatrix} F_1' \\ F_2' \\ F_3' \end{pmatrix} = J^T \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

Using this notation, we can derive the invariance of div $\vec{F}$ and curl $\vec{F}$ under an orthogonal change of coordinates.

a) Scalar field: div $\vec{F}$ = (in new basis) $\frac{\partial F_1'}{\partial x'} + \frac{\partial F_2'}{\partial y'} + \frac{\partial F_3'}{\partial z'}$

divergence in new basis → (tensor notation) $= \frac{\partial F_i'}{\partial x_i'} = \frac{\partial}{\partial x_i'}(J_{ji} F_j)$ (transform components: F_i' i.t.o. F_j)

$$= J_{ji} \frac{\partial}{\partial x_i'} F_j = J_{ji} \frac{\partial F_j}{\partial x_k} \underbrace{\frac{\partial x_k}{\partial x_i'}}_{J_{ki}} \quad \text{(chain rule)}$$

$$= J_{ji} J_{ki} \frac{\partial F_j}{\partial x_k} = J_{ji} J^T_{ik} \frac{\partial F_j}{\partial x_k} = (J^T J)_{jk} \frac{\partial F_j}{\partial x_k}$$

J is orthogonal ↓

$$= \delta_{jk} \frac{\partial F_j}{\partial x_k} = \frac{\partial F_j}{\partial x_j} \leftarrow \text{divergence in old basis.}$$

b) Vector field: curl $\vec{F} = \hat{e}_i' (\text{curl}'\vec{F})_i = \hat{e}_i' (\nabla' \times \vec{F})_i$

This problem is rather tricky in tensor notation, as it involves a new identity:

$$\boxed{\varepsilon_{ijk} J_{li} J_{mj} J_{pk} = \varepsilon_{lmp}}$$

(I guess the alternative, writing out the components, is worse...)

$$= \hat{e}_i' \varepsilon_{ijk} \frac{\partial}{\partial x_j'} F_k'$$

$\left[\hat{e}_i' = (\hat{e}_i' \cdot \hat{e}_l)\hat{e}_l = J_{li} \hat{e}_l \right.$ expansion of $\hat{e}_i'$ in old basis

$$= \varepsilon_{ijk} J_{li} \hat{e}_l \frac{\partial F_k'}{\partial x_j'}$$

$\left[F_k' = J_{pk} F_p \right.$ components of F transform

$$= \varepsilon_{ijk} J_{li} J_{pk} \hat{e}_l \frac{\partial F_p}{\partial x_j'}$$

$\left[\frac{\partial f}{\partial x_j'} = \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial x_j'} = J_{mj} \frac{\partial f}{\partial x_m} \right.$ chain rule

$$= \underbrace{\varepsilon_{ijk} J_{li} J_{mj} J_{pk}}_{\varepsilon_{lmp}} \hat{e}_l \frac{\partial F_p}{\partial x_m}$$

This expression equals curl $\vec{F}$ in the old coordinate system,

curl $\vec{F} = \varepsilon_{lmp} \hat{e}_l \frac{\partial F_p}{\partial x_m}$, provided $\varepsilon_{ijk} J_{li} J_{mj} J_{pk} = \varepsilon_{lmp}$.

← l^{th} row of transformation matrix

But in fact this is true: For fixed l, J_{li} $(i=1,2,3)$ gives the components of $\hat{e}_l$ (l^{th} old basis vector) written in the new basis; similarly for J_{mj}, J_{pk}.

Thus $\varepsilon_{ijk} J_{li} J_{mj} J_{pk}$ is the determinant of the matrix $\begin{pmatrix} \cdots \hat{e}_l \cdots \\ \cdots \hat{e}_m \cdots \\ \cdots \hat{e}_p \cdots \end{pmatrix}$ (coordinates w.r.t. new basis), which equals the scalar triple product $\hat{e}_l \cdot (\hat{e}_m \times \hat{e}_p) \overset{\text{check.}}{=} \varepsilon_{lmp}$

Solution to Section 5.8 # 10

Yes, the Laplacian is invariant under linear orthogonal transformations. This follows immediately from the fact that the gradient and divergence are invariant under such transformations (for the gradient, this was shown in class and in the textbook; for divergence, this was problem 8 of Section 5.8).

We can also directly show the invariance, as an exercise in the chain rule.

We need to show $\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} + \frac{\partial^2 f}{\partial z'^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

I will first compute some terms directly, to give the idea:

By the chain rule, $\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x'}$

Recall $J_{ij} = \hat{e}_i \cdot \hat{e}'_j = \frac{\partial x_i}{\partial x'_j}$

$$= J_{11}\frac{\partial f}{\partial x} + J_{21}\frac{\partial f}{\partial y} + J_{31}\frac{\partial f}{\partial z}$$

[J_{ij} constant for a linear transformation]

$$\Rightarrow \frac{\partial^2 f}{\partial x'^2} = J_{11}\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\right) + J_{21}\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial y}\right) + J_{31}\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial z}\right)$$

where $\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial x'} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial x'} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial x'} = J_{11}\frac{\partial^2 f}{\partial x^2} + J_{21}\frac{\partial^2 f}{\partial x \partial y} + J_{31}\frac{\partial^2 f}{\partial x \partial z}$,

Similarly $\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial y}\right) = J_{11}\frac{\partial^2 f}{\partial x \partial y} + J_{21}\frac{\partial^2 f}{\partial y^2} + J_{31}\frac{\partial^2 f}{\partial z \partial y}$, etc.

Thus compute all second partial derivatives using the chain rule (27 terms!), and add. In interpreting the answer, note that

coeff. of $\partial^2 f/\partial x^2 \rightarrow J_{11}J_{11} + J_{12}J_{12} + J_{13}J_{13} = J_{11}J^T_{11} + J_{12}J^T_{21} + J_{13}J^T_{31} = (JJ^T)_{11} = 1$

of $2\frac{\partial^2 f}{\partial x \partial y} \rightarrow J_{11}J_{21} + J_{12}J_{22} + J_{13}J_{23} = (JJ^T)_{12} = 0$ etc, since $JJ^T = I$.

This calculation is much simpler using tensor notation:

$$\frac{\partial}{\partial x'_i}\frac{\partial f}{\partial x'_i} = \frac{\partial}{\partial x'_i}\left(\frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x'_i}\right) = \frac{\partial x_j}{\partial x'_i}\left(\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right)\frac{\partial x_k}{\partial x'_i}\right) = \frac{\partial x_k}{\partial x'_i}\frac{\partial x_j}{\partial x'_i}\frac{\partial^2 f}{\partial x_k \partial x_j} = \delta_{kj}\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j}$$

new coords ↑ ; $J_{ki}J_{ji} = (JJ^T)_{kj} = \delta_{kj}$; old ↑