MATH 252-3 **Spring 2004**

Vector Calculus

Homework Set 2 Due Wednesday, 21 January 2004

Course Web Site: `http://www.math.sfu.ca/~ralfw/math252/`

Problems from Davis and Snider "Introduction to Vector Analysis":

- Section 1.7 (pp.23–24): 3
- Section 1.8 (pp.29–30): 7, 8, 11, 12, 18
- Section 1.9 (pp.34–35): 20, 24, 25, 28
- Section 1.10 (pp.38–40): 5, 8, 9, 12, 18, 28
- Section 5.7 (pp.315–316): 15

Also, please read the notes and do the *Additional Problems* on the accompanying pages.

(. . . questions on next page . . .)

1. *Abstract Vector Spaces and Inner Products*
Let V be a *vector space* over the real numbers $\mathbb{R}$, containing elements $\mathbf{A}$, $\mathbf{B}\dots$ called "vectors". This means that the operations of vector addition and scalar multiplication are defined, satisfying the properties:

   - V is closed under addition and scalar multiplication:
   If $\mathbf{A}$, $\mathbf{B} \in V$, s, $t \in \mathbb{R}$, then $s\mathbf{A} + t\mathbf{B} \in V$;
   - Addition and scalar multiplication are commutative and associative, and scalar multiplication is distributive over addition.

   An *inner product* for a real vector space satisfies the following properties:

   - $\mathbf{A} \cdot \mathbf{B} \in \mathbb{R}$ for $\mathbf{A}$, $\mathbf{B} \in V$
   - $\mathbf{A} \cdot \mathbf{A} \geq 0$ for $\mathbf{A} \in V$, and $\mathbf{A} \cdot \mathbf{A} = 0$ if and only if $\mathbf{A} = \mathbf{0}$.
   - $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
   - $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \quad (s\mathbf{A}) \cdot \mathbf{B} = s(\mathbf{A} \cdot \mathbf{B})$ for $s \in \mathbb{R}$

   Given an inner product, we can define the *norm* (magnitude) of a vector via

   $$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

   (the norm is sometimes written as $\|\mathbf{A}\|$ to distinguish it from the absolute value of a real or complex number).

   (a) Using just the above properties of V and the inner product, show the *Cauchy-Schwarz inequality*,

   $$|\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}|\,|\mathbf{B}| \ .$$

   [Hint: First show that the inequality holds (as an equality) if either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. For $\mathbf{B} \neq \mathbf{0}$, observe that $|\mathbf{A} + \lambda\mathbf{B}|^2 \geq 0$ for every $\lambda \in \mathbb{R}$. Write this as $(\mathbf{A} + \lambda\mathbf{B}) \cdot (\mathbf{A} + \lambda\mathbf{B})$ and multiply it out; now substitute, successively, $\lambda = |\mathbf{A}|/|\mathbf{B}|$ and $\lambda = -|\mathbf{A}|/|\mathbf{B}|$ to obtain the Cauchy-Schwarz inequality.]

   (b) Use the Cauchy-Schwarz inequality to prove the *triangle inequality*,

   $$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| \ .$$

   [Hint: Square both sides, noting that the left-hand side becomes $|\mathbf{A} + \mathbf{B}|^2 = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$; multiply this out and use the Cauchy-Schwarz inequality.]

   (c) Prove the reverse triangle inequality

   $$\big|\,|\mathbf{A}| - |\mathbf{B}|\,\big| \leq |\mathbf{A} - \mathbf{B}| \ .$$

   [Hint: Apply the triangle inequality to $\mathbf{A} = (\mathbf{A} - \mathbf{B}) + \mathbf{B}$; then reverse the roles of $\mathbf{A}$ and $\mathbf{B}$.]

   Comments: You should check that all the above definitions and results make sense in terms of what you have already learnt about vectors, the dot product and their geometric interpretation.

   The point of these rather abstract calculations is that these important inequalities depend only on general properties of vector spaces and the inner product. In class we established these inequalities using the geometric properties of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ — in particular, we defined the inner (scalar/dot) product by $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$, so that the Cauchy-Schwarz inequality follows from $|\cos\theta| \leq 1$. The present calculations give an alternative derivation, and show that the same inequalities must hold for other vector spaces with appropriate inner products defined, for instance for $\mathbb{R}^n$.

2. *Linear Independence, the Gram-Schmidt Process, and Orthogonal Transformations*
A set of nonzero vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is *linearly independent* if no nontrivial linear combination of the vectors vanishes; that is, if the vectorial equation

$$\sum_{i=1}^{n} c_i \mathbf{a}_i = \mathbf{0}$$

has the unique solution $c_1 = c_2 = \cdots = c_n = 0$. Equivalently, none of the vectors can be expressed as a linear combination of the others; for instance, there are no solutions of $\mathbf{a}_n = c'_1\mathbf{a}_1 + \cdots + c'_{n-1}\mathbf{a}_{n-1}$. For vectors in $\mathbb{R}^3$, linear independence of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ means that the three vectors are not coplanar, that is, that $\mathbf{a}_3$ does not lie in the plane defined by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (assumed non-parallel).

Given a set of linearly independent vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, we may successively construct a set of mutually orthogonal vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ (spanning the same subspace) using the *Gram-Schmidt process*. The idea is as follows:

- Choose $\mathbf{b}_1 = \mathbf{a}_1$ (or any nonzero multiple of $\mathbf{a}_1$).
- Next, we wish to choose $\mathbf{b}_2$ as a vector in the subspace spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, but orthogonal to $\mathbf{b}_1$ (consequently, $\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\} = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\}$). Thus define

  $$\mathbf{b}_2 = \mathbf{a}_2 - \lambda \mathbf{b}_1 \ ,$$

  and find λ using the condition $\mathbf{b}_2 \cdot \mathbf{b}_1 = 0$; this gives $\lambda = \mathbf{a}_2 \cdot \mathbf{b}_1 / \mathbf{b}_1 \cdot \mathbf{b}_1$, so that

  $$\mathbf{b}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = \mathbf{a}_2 - \operatorname{proj}_{\mathbf{b}_1}\mathbf{a}_2.$$

  Note that $\mathbf{b}_2 \neq \mathbf{0}$, since if it were zero that would imply that $\mathbf{a}_2$ and $\mathbf{b}_1 = \mathbf{a}_1$ were linearly dependent, which they are not.
- Now we can proceed analogously to obtain the remaining orthogonal vectors: At the kth step in the process ($k \geq 2$), we obtain $\mathbf{b}_k$ by taking $\mathbf{a}_k$ and subtracting from it its projection onto the subspace spanned by the previously constructed vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}$:

  $$\mathbf{b}_k = \mathbf{a}_k - \frac{\mathbf{a}_k \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 - \cdots - \frac{\mathbf{a}_k \cdot \mathbf{b}_{k-1}}{\mathbf{b}_{k-1} \cdot \mathbf{b}_{k-1}}\mathbf{b}_{k-1} = \mathbf{a}_k - \sum_{i=1}^{k-1} \frac{\mathbf{a}_k \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}\mathbf{b}_i \ .$$

  You should verify that this indeed implies that $\mathbf{b}_k \cdot \mathbf{b}_i = 0$ for $i = 1, \ldots, k-1$, and that $\mathbf{b}_k \neq \mathbf{0}$ since $\mathbf{a}_k \notin \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}\} = \operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}$.
  Once the orthogonal set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ has been constructed recursively, it is straightforward to obtain an *orthonormal* set of vectors $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ spanning the same subspace, by defining $\mathbf{e}'_k = \mathbf{b}_k / |\mathbf{b}_k|$, $k = 1, \ldots, n$.

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In this problem, we will use the Gram-Schmidt process to construct an orthonormal basis of $\mathbb{R}^3$ from a set of three independent vectors, and then compute the coordinates of given vectors $\mathbf{A}$ and $\mathbf{B}$ with respect to the new basis:

Consider the vectors in $\mathbb{R}^3$, expressed in terms of the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{a}_1 = \mathbf{i} - \mathbf{j}\ , \quad \mathbf{a}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}\ , \quad \mathbf{a}_3 = 2\mathbf{j} + \mathbf{k}.$$

Also define the vectors $\mathbf{A}$ and $\mathbf{B}$ in this basis as

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \quad \mathbf{B} = \mathbf{i} - \mathbf{k},$$

where the components may be found as $A_i = \mathbf{A}\cdot\mathbf{e}_i$, $B_i = \mathbf{B}\cdot\mathbf{e}_i$, $i = 1, \ldots, 3$. (For simplicity, you may wish to represent the vectors as column vectors containing the components, provided you are clear which basis is being used...)

(a) Show that the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent. Use the Gram-Schmidt process to obtain a set of three mutually orthogonal vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ from $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, where $\mathbf{b}_1 = \mathbf{a}_1$. Also normalize the vectors $\mathbf{b}_i$ to obtain a new orthonormal basis $\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\} = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ for $\mathbb{R}^3$, where $\mathbf{e}'_i$ is a unit vector in the direction of $\mathbf{b}_i$, $i = 1, \ldots, 3$.

(b) Use orthogonal projections onto the new basis vectors to find the components A'_i, B'_i of $\mathbf{A}$ and $\mathbf{B}$ with respect to the new basis, where

$$\mathbf{A} = \sum_{i=1}^{3} A'_i \mathbf{e}'_i$$

and similarly for $\mathbf{B}$.

(c) By taking appropriate inner products of the old and new basis vectors, compute the transformation matrix (Jacobian matrix) J of the coordinate transformation, and the transpose J^T. Compute the matrix product to verify that $J^T J = I$.

(d) Now compute the components A'_i, B'_i of $\mathbf{A}$ and $\mathbf{B}$ with respect to the new basis by using the transformation matrix J, using $A'_i = \sum_{j=1}^{3} J_{ji} A_j$; your answers should be the same as obtained in part (b). Next, use the matrix J and the new coordinates A'_i, B'_i to recompute the old coordinates A_i, B_i, and verify that you have found the values you started with.

(e) Lastly, we will confirm that the dot product $\mathbf{A} \cdot \mathbf{B}$ defines a scalar: compute $\mathbf{A} \cdot \mathbf{B}$ in both the old and the new coordinate system, and verify that your answers are the same.