MATH 252-3 **Spring 2004**

Vector Calculus

Homework Set 7 Due Wednesday, 3 March 2004

Course Web Site: `http://www.math.sfu.ca/~ralfw/math252/`

Problems from Davis and Snider "Introduction to Vector Analysis":

- Section 3.10 (pp.169–170): 2, 8, 9, 10, 12, 13, 16 (use the result of problem 2)
- Section 5.8 (pp.326–329): 8, 10

Notes:

1. Compare the ease of calculating the divergence of the inverse-square force field $\mathbf{F}(\mathbf{R}) = \mathbf{R}/R^3$ in spherical coordinates (problem 13 of Section 3.10, for $n = -2$) with the same calculation in Cartesian coordinates (problem 4 of Section 3.4; see note 1 of Homework Set 5).

Extra problems:

1. *Maxwell's Equations and Electromagnetic Waves*
   In this problem, we will demonstrate the usefulness of vector identities by investigating the differential form of *Maxwell's Equations*, which form the basis of the theory of electromagnetism: their solutions describe the phenomenology of electrostatics, magnetostatics, and dynamic phenomena such as electromagnetic waves.

   Maxwell's equations are a system of linear partial differential equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. In SI units (MKS units: that is, using the metre, kg, second, and Ampere as basic units), in the absence of magnetic or polarizable media

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

   Here ρ is the charge density (charge per unit volume: units $\mathrm{C/m^3}$) and $\mathbf{J}$ is the current density (current per unit area: units $\mathrm{A/m^2}$, where C = Coulomb, A = Ampere = Coulomb/second). Also, ϵ_0 = permittivity of free space $= 8.8542 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$, (N = Newtons $= \mathrm{kg\,m\,s^{-2}}$) and μ_0 = permeability of free space $= 4\pi \times 10^{-7}\mathrm{NA^{-2}}$.

   The first equation (1) above is the differential form of Gauss' Law for electricity; (2) is Gauss' Law for magnetism (no magnetic monopoles); (3) represents Faraday's Law of induction, while equation (4) is Ampère's Law. [In dielectric, magnetic or polarizable materials, Maxwell's equations are modified to take into account the electric permittivity, magnetic permeability and polarization of the medium.]

   The mathematical and physical consequences of Maxwell's equations are profound, and represent a major scientific achievement of the 19th century. For now, we will explore how vector identities can be used to deduce the conservation of charge and the existence of electromagnetic waves, and study some basic properties of these waves.

(a) From Maxwell's equations (1) and (4), derive the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

which represents the conservation of charge (where ρ is charge density, and $\mathbf{J}$ is the current density, or charge flux density).
[Hint: Take $\partial/\partial t$ of (1); use the fact that partial derivatives with respect to space and time variables commute (equality of mixed partial derivatives), so that $\partial/\partial t(\nabla \cdot \mathbf{E}) = \nabla \cdot (\partial \mathbf{E}/\partial t)$; substitute for $\partial \mathbf{E}/\partial t$ in (4) and use a vector identity.]

(b) Maxwell's equations in a vacuum, in the absence of any charges or currents ($\rho = 0$, $\mathbf{J} = \mathbf{0}$), are given by:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 && (5)\\ \nabla \cdot \mathbf{B} &= 0 && (6)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (7)\\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && (8) \end{aligned}$$

Show that the electric field $\mathbf{E}$ satisfies the *wave equation*

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E} \,, \tag{9}$$

where $c^2 = 1/\mu_0\epsilon_0$. Similarly, verify that the magnetic field $\mathbf{B}$ also satisfies (9).
(This vector equation means that each *component* E_i, B_i of $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation $\partial^2 u/\partial t^2 = c^2 \nabla^2 u$.)

[Hint: Take $\partial/\partial t$ of (8), commute the spatial and temporal derivatives, and substitute from (7); use a suitable vector identity.]

The constant $c = 1/\sqrt{\mu_0 \epsilon_0} \approx 2.99 \times 10^8 \mathrm{m\,s^{-2}}$ is the speed of light! By this calculation, Maxwell realized that *light is an electromagnetic wave, propagating at speed c.*

We will explore some of the details of Maxwell's calculation. To see why (9) is a wave equation, assume single-frequency solutions to (5)–(8) for the electric and magnetic field of the form

$$\begin{aligned} \mathbf{E} &= [A\cos(\omega t - \mathbf{k}\cdot\mathbf{R}) + B\sin(\omega t - \mathbf{k}\cdot\mathbf{R})]\mathbf{u}_E \,, && (10)\\ \mathbf{B} &= [C\cos(\omega t - \mathbf{k}\cdot\mathbf{R}) + D\sin(\omega t - \mathbf{k}\cdot\mathbf{R})]\mathbf{u}_B \,. && (11) \end{aligned}$$

Here: $\mathbf{u}_E$ and $\mathbf{u}_B$ are suitable constant unit vectors indicating the *direction* of the fields $\mathbf{E}$ and $\mathbf{B}$, respectively;
$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the position vector;
ω is a constant, the angular frequency of the wave ($\omega = 2\pi\nu$, where ν is the *frequency*);
$\mathbf{k} = k_1\hat{\mathbf{i}} + k_2\hat{\mathbf{j}} + k_3\hat{\mathbf{k}}$ is a constant vector, whose direction indicates the direction of propagation of the wave, and whose magnitude is $|\mathbf{k}| = 2\pi/\lambda$, where λ is the *wavelength*;
and A, B, C and D are constants.
[Note: you should be careful to distinguish between $\mathbf{k}$, the wave propagation vector with magnitude $2\pi/\lambda$, and $\hat{\mathbf{k}}$, the unit vector parallel to the z-axis.]

(c) Show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equation (9) provided $c = \omega/|\mathbf{k}|$.

[You need consider only $\mathbf{E}$. Compute $\partial^2\mathbf{E}/\partial x^2$ to show that $\partial^2\mathbf{E}/\partial x^2 = -k_1^2\mathbf{E}$; by a similar calculation for the y and z derivatives, show that $\nabla^2\mathbf{E} = -|\mathbf{k}|^2\mathbf{E}$. Similarly, compute $\partial^2\mathbf{E}/\partial t^2$, and show $\partial^2\mathbf{E}/\partial t^2 = -\omega^2\mathbf{E}$. Thus show that the expression (10) for $\mathbf{E}$ satisfies the wave equation (9) provided ω and $|\mathbf{k}|$ are related by $\omega = c|\mathbf{k}|$.]

(d) Explain why the fields $\mathbf{E}$ and $\mathbf{B}$ defined in (10)–(11) represent *waves* propagating in the direction of $\mathbf{k}$, with *wave speed* $c = \nu\lambda = \omega/|\mathbf{k}|$.

[You need consider only $\mathbf{E}$. Show that $\mathbf{E}$ takes the same value at time $t = 0$, position vector $\mathbf{R}_0$, and at time t, position vector $\mathbf{R}$, provided that $\omega t - \mathbf{k}\cdot(\mathbf{R}-\mathbf{R}_0) = 0$. Thus show that if $\mathbf{R}-\mathbf{R}_0$ is parallel to $\mathbf{k}$, then $\mathbf{E}$ is constant along points with position vectors $\mathbf{R} = \mathbf{R}(t)$ satisfying $|\mathbf{R}-\mathbf{R}_0| = \omega t/|\mathbf{k}| = ct$; and hence interpret c as the wave speed.]

(e) Show that $\nabla\cdot\mathbf{E} = 0$ implies $\mathbf{k}\cdot\mathbf{u}_E = 0$. Thus show that (5) and (10) imply that that the electric field vector $\mathbf{E}$ is perpendicular to the direction of propagation, $\mathbf{k}\cdot\mathbf{E} = 0$. Similarly, show that $\mathbf{k}$ is perpendicular to $\mathbf{u}_B$ and thus to $\mathbf{B}$.

(f) Substitute $\mathbf{E}$ and $\mathbf{B}$ into (7) to show that $\mathbf{k}\times\mathbf{u}_E$ is parallel to $\mathbf{u}_B$, and thus that $\mathbf{u}_E\cdot\mathbf{u}_B = 0$; thus the electric and magnetic fields are mutually perpendicular.
Also use (7) to find a relation between A and C, and a relation between B and D. (You would obtain the same result using (8) instead of (7)).

In summary, for an electromagnetic wave, such as light (or radio waves, infrared or ultraviolet radiation, X-rays or gamma rays) $\mathbf{k}$, $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular vectors; the changing electric field induces a changing magnetic field (according to (8)) which in turn induces an electric field (by (7)), and together these fields propagate at speed c in the direction of $\mathbf{k}$ while remaining mutually perpendicular to each other and to $\mathbf{k}$.

For more information on the theory of electromagnetism, see Appendix D of the textbook by Davis and Snider, the book by Shey in the library reserves, or any one of many suitable physics references.

2. *Vector Differential Operators and Linear Orthogonal Transformations*
Consider the scalar and vector fields (expressed in Cartesian coordinates relative to the standard basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\} = \{\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}\}$)

$$\begin{aligned} f(x,y,z) &= x^2 + yz + z^2\ , \\ \mathbf{G}(x,y,z) &= G_1(x,y,z)\hat{\mathbf{i}} + G_2(x,y,z)\hat{\mathbf{j}} + G_3(x,y,z)\hat{\mathbf{k}} = \hat{\mathbf{i}} + (y-2x)\hat{\mathbf{j}} - 2xz\hat{\mathbf{k}}. \end{aligned}$$

(a) Compute **grad** f, div $\mathbf{G}$, **curl** $\mathbf{G}$ and $\nabla^2 f$ (in the $\{\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}\}$ coordinate system).

We will now consider a linear orthogonal transformation to a new Cartesian coordinate system. Define the new basis vectors $\{\mathbf{e}_1',\mathbf{e}_2',\mathbf{e}_3'\} = \{\hat{\mathbf{i}}',\hat{\mathbf{j}}',\hat{\mathbf{k}}'\}$ (with respect to the old coordinate system) by

$$\hat{\mathbf{i}}' = \frac{1}{2}(\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{k}}), \quad \hat{\mathbf{j}}' = \frac{1}{2}(-\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{k}}), \quad \hat{\mathbf{k}}' = -\hat{\mathbf{j}}. \tag{12}$$

This new basis is obtained by first rotating the $\{\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}\}$ basis by 90° counterclockwise about the x-axis, and then rotating by 60° counterclockwise about the (new) z-axis.

(b) Verify that $\{\hat{\mathbf{i}}',\hat{\mathbf{j}}',\hat{\mathbf{k}}'\}$ forms a right-handed orthonormal coordinate system, that is, the vectors are pairwise mutually orthogonal, have norm 1, and $\hat{\mathbf{i}}'\times\hat{\mathbf{j}}' = \hat{\mathbf{k}}'$. Also write down the transformation matrix (Jacobian matrix) J for the coordinate transformation, and its transpose J^T.

(c) Invert the formulas (12), that is, write the old basis vectors $\{\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}\}$ in terms of the new basis vectors $\{\hat{\mathbf{i}}',\hat{\mathbf{j}}',\hat{\mathbf{k}}'\}$.

(d) Consider a point P whose position vector $\mathbf{R}$ has coordinates (x, y, z) with respect to the old basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, and coordinates (x', y', z') with respect to the new basis $\{\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'\}$; that is,

$$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}' + z'\hat{\mathbf{k}}'.$$

Write the formulas for the old coordinates (x, y, z) in terms of the new coordinates (x', y', z').

Now we will obtain the expressions for the scalar field f and the vector field $\mathbf{G}$ in terms of the new coordinate system. Note that the fields stay the same; for instance, at a given point P, the value of f is unchanged. However, the coordinates of the point change as we change the basis, and thus the *formula* for f will change.

(e) Express the function $f(x, y, z)$ in the new coordinate system, as $f'(x', y', z')$, by substituting the expressions for the old coordinates in terms of the new.
[Answer: $f'(x', y', z') = x'^2 + y'^2 - \frac{\sqrt{3}}{2}x'z' - \frac{1}{2}y'z'$.]

(f) Express the vector field $\mathbf{G}$ with respect to the new basis (you will need to find the components $G'_i(x', y', z')$, $i = 1, 2, 3$ by expressing the old coordinates in terms of the new, and also by finding the new components G'_i in terms of the old components G_i).
[Answer:

$$\mathbf{G} = \left[\frac{1}{2} - \frac{3}{4}x'^2 + \frac{\sqrt{3}}{2}x'y' + \frac{3}{4}y'^2\right]\hat{\mathbf{i}}'$$
$$+ \left[-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}x'^2 + \frac{1}{2}x'y' + \frac{\sqrt{3}}{4}y'^2\right]\hat{\mathbf{j}}' + \left[x' - \sqrt{3}y' + z'\right]\hat{\mathbf{k}}' \ . \quad]$$

Let us now compute the vector differential operators in the new coordinate system:

(g) Compute **grad** f in the new coordinate system; that is,

$$\mathbf{grad}\ f = \frac{\partial f'}{\partial x'}\hat{\mathbf{i}}' + \frac{\partial f'}{\partial y'}\hat{\mathbf{j}}' + \frac{\partial f'}{\partial z'}\hat{\mathbf{k}}'.$$

[Answer: $\mathbf{grad}\ f = \left[2x' - \frac{\sqrt{3}}{2}z'\right]\hat{\mathbf{i}}' + \left[2y' - \frac{1}{2}z'\right]\hat{\mathbf{j}}' - \left[\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right]\hat{\mathbf{k}}'$.]

(h) Compute div $\mathbf{G}$ in the new coordinate system.
[Answer: div $\mathbf{G} = 1 - x' + \sqrt{3}y'$.]

(i) Compute **curl G** in the new coordinate system.
[Answer: $\mathbf{curl\ G} = -\sqrt{3}\,\hat{\mathbf{i}}' - \hat{\mathbf{j}}' - \left[\sqrt{3}x' + y'\right]\hat{\mathbf{k}}'$.]

(j) Compute the Laplacian $\nabla^2 f$ in the new coordinate system.
[Answer: 4.]

Lastly, we will verify that the expressions for div $\mathbf{G}$ and $\nabla^2 f$ represent the same scalar field, and **grad** f and **curl G** the same vector field, in the old and new coordinate systems; that is, that **grad**, div, and **curl** truly represent *vector* operations (coordinate-free). This is of course consistent with the fact that these operations were defined without reference to a coordinate system: **grad** f gives the maximum rate of change of f with respect to distance, in the direction of that maximum; div $\mathbf{G}$ is the flux per unit volume, and the component of **curl G** in a particular direction is the circulation per unit area for a surface with normal in that direction.

(k) Transform the fields **grad** f, div $\mathbf{G}$ and **curl G** from part (a) (which were obtained by first differentiating with respect to the old coordinate system) to the new basis $\{\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'\}$ with coordinates (x', y', z'), and confirm that the answers agree with those of parts (g), (h) and (i) (which were found by first transforming to the new basis, and then differentiating in the new coordinate system).