MATH 252-3 **Spring 2004**

Vector Calculus

Homework Set 8 Due Wednesday, 10 March 2004

Course Web Site: `http://www.math.sfu.ca/~ralfw/math252/`

Problems from Davis and Snider "Introduction to Vector Analysis":

- Section 3.7 (pp.145–146): 2, 3, 5, 7
- Section 3.11 (pp.180–182): 3, 7, 9, 11, 12, 13, 14

Extra problems:

1. *Two-variable second derivative test*

   (a) Consider a scalar field on $\mathbb{R}^2$, that is, a function of two variables $f(x, y)$. Using the Hessian matrix, we saw that a point $\mathbf{R}_0 = x_0\mathbf{i} + y_0\mathbf{j}$ is a local minimum of f if $\mathbf{R}_0$ is a critical point and $\nabla\nabla f(\mathbf{R}_0)$ is positive definite, that is,

   $$\nabla f = \mathbf{0}, \qquad \text{and} \qquad \mathbf{h}\cdot\nabla\nabla f(\mathbf{R}_0)\cdot\mathbf{h} > 0 \quad \text{for all} \quad \mathbf{h} \neq \mathbf{0}.$$

   Write out the above inequality in terms of the components h_1, h_2 of $\mathbf{h}$ and of the Hessian matrix (or dyadic). By suitable choices of h_1 and h_2, hence derive the second derivative minimum test for a function of two variables: that $\mathbf{R}_0$ is a (strict) local minimum of f provided

   $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0, \quad \frac{\partial^2 f}{\partial x^2} > 0, \quad D \equiv \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0,$$

   with all derivatives evaluated at (x_0, y_0). What is the corresponding condition for a (strict) local maximum, and for a saddle point?

   (b) Find the critical points of the following functions, and then determine whether they are local maxima, local minima, or saddle points:

   $$\begin{array}{ll} \text{(i)} & f(x,y) = 3x^2 + 2xy + 2x + y^2 + y + 4 \\ \text{(ii)} & f(x,y) = (x+y)(xy+1) \end{array}$$