MATH 252-3 **Spring 2004**

Vector Calculus

Homework Set 9 Due Wednesday, 17 March 2004

Course Web Site: `http://www.math.sfu.ca/~ralfw/math252/`

Problems from Davis and Snider "Introduction to Vector Analysis":

- Section 4.1 (pp.190–192): 3, 6, 10, 12, 14
- Section 4.2 (p.196): 1, 5
- Section 4.3 (p.204): 2(c), 3(c), 4, 5, 6
- Section 4.4 (pp.212–213): 1(c,d), 2, 7, 9

Extra problem:

1. *Conservative Forces*
If a *force field* $\mathbf{F}$ is conservative, in physics it is customary to choose the opposite sign for the potential function, and write $\mathbf{F} = -\nabla\phi$ (or frequently $\mathbf{F} = -\nabla V$); the sign convention implies that the force $\mathbf{F}$ acts in the direction of *decreasing* potential ϕ.

   (a) Show that the work done by a conservative force $\mathbf{F}$ in moving a particle of mass m from point P to Q, is path-independent, and is given by

   $$W = \int_P^Q \mathbf{F} \cdot d\mathbf{R} = \phi(P) - \phi(Q).$$

   (b) Using Newton's Law $\mathbf{F} = m\mathbf{a} = m\mathbf{v}'$, show that the work done in moving a particle from P to Q along the path $C : \mathbf{R} = \mathbf{R}(t)$, $a \le t \le b$, equals the change in the kinetic energy $\frac{1}{2}mv^2$ (where the velocity is $\mathbf{v}(t) = \mathbf{R}'(t)$, and the speed is $v = |\mathbf{v}|$).

   (c) By combining the above two expressions for the work done, conclude that

   $$\tfrac{1}{2}mv_P^2 + \phi(P) = \tfrac{1}{2}mv_Q^2 + \phi(Q).$$

   The scalar field $\phi(\mathbf{R})$ thus clearly has units of energy, and is called the *potential energy*; the work done by $\mathbf{F}$ is the *decrease* in potential energy. The above equation then expresses the *Law of Conservation of Energy*: The total energy, the sum of the *kinetic energy* $\frac{1}{2}mv^2$ and the potential energy $\phi(\mathbf{R})$, remains constant under the action of a conservative force.

   (d) Show that the gravitational force $\mathbf{F} = -GMm\mathbf{R}/|\mathbf{R}|^3$ is conservative, find the associated potential energy (it is easiest to use spherical coordinates), and write an expression for the total energy. Show that the work done in moving a particle of mass m from radius $R = |\mathbf{R}| = R_0$ to $R = R_1$ is

   $$W = GMm\left(\frac{1}{R_0} - \frac{1}{R_1}\right);$$

   if the corresponding initial and final speeds are $v = v_0$ and $v = v_1$, write down the equation of energy conservation and use it to find v_0 in terms of R_0, R_1 and v_1. For a particle leaving the surface of the earth (radius R_0, mass M) with speed v_0, hence show that if $v_0 < v_e \equiv \sqrt{2GM/R_0}$, then the maximum height R_1 reached (when $v_1 = 0$) is finite, while if $v_0 \ge v_e$, then the particle can escape the earth's gravitational influence (we can have $R_1 \to \infty$); the speed v_e is called the *escape velocity* (of course we have neglected effects such as air resistance).