Math 252
Homework 2 Additional Problems

Some Solutions.

1. Abstract Vector Spaces and Inner Products

The definitions of a general vector space and inner product are given on the problem set; please review these.

a) Cauchy-Schwarz inequality: $|\vec{A}\cdot\vec{B}| \le |\vec{A}|\,|\vec{B}|$

absolute value of a scalar — magnitudes of vectors

Proof

- If $\vec{A}=\vec{0}$ or $\vec{B}=\vec{0}$, then $\vec{A}\cdot\vec{B}=0$, $|\vec{A}|\,|\vec{B}|=0$
- Suppose $\vec{B}\neq\vec{0}$.

As given in the problem, notice that $|\vec{A}+\lambda\vec{B}|^2 \ge 0 \quad \forall\lambda\in\mathbb{R}$

Multiplying out:

$$0 \le |\vec{A}+\lambda\vec{B}|^2 = (\vec{A}+\lambda\vec{B})\cdot(\vec{A}+\lambda\vec{B})$$
$$= \vec{A}\cdot\vec{A} + 2\lambda\,\vec{A}\cdot\vec{B} + \lambda^2\,\vec{B}\cdot\vec{B}$$
$$= |\vec{A}|^2 + 2\lambda\,\vec{A}\cdot\vec{B} + \lambda^2|\vec{B}|^2.$$

This inequality holds for all λ, so it holds in particular for $\lambda = |\vec{A}|/|\vec{B}|$. Substituting:

$$0 \le |\vec{A}|^2 + 2\frac{|\vec{A}|}{|\vec{B}|}\vec{A}\cdot\vec{B} + \frac{|\vec{A}|^2}{|\vec{B}|^2}|\vec{B}|^2 = 2|\vec{A}|^2 + 2\frac{|\vec{A}|}{|\vec{B}|}\vec{A}\cdot\vec{B}$$

$$\Rightarrow \quad -2\frac{|\vec{A}|}{|\vec{B}|}\vec{A}\cdot\vec{B} \le 2|\vec{A}|^2$$

$$\Rightarrow \quad -\vec{A}\cdot\vec{B} \le |\vec{A}|\,|\vec{B}|. \qquad ①$$

Similarly, substituting $\lambda = -|\vec{A}|/|\vec{B}|$, we find

$$0 \le 2|\vec{A}|^2 - 2\frac{|\vec{A}|}{|\vec{B}|}\vec{A}\cdot\vec{B} \quad \Rightarrow \quad \vec{A}\cdot\vec{B} \le |\vec{A}|\,|\vec{B}| \qquad ②$$

Combining ① and ②, we obtain the result, $|\vec{A}\cdot\vec{B}| \le |\vec{A}|\,|\vec{B}|$. ☒

b) Triangle inequality: $|\vec{A}+\vec{B}| \le |\vec{A}| + |\vec{B}|$ (using Cauchy-Schwarz)

Proof.

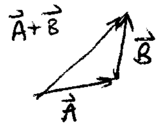
Since the l.h.s. and r.h.s. are both non-negative, by squaring both sides, it is sufficient to show that

$$|\vec{A}+\vec{B}|^2 \le (|\vec{A}| + |\vec{B}|)^2.$$

But $\quad |\vec{A}+\vec{B}|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = \vec{A}\cdot\vec{A} + 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$

$$= |\vec{A}|^2 + 2\vec{A}\cdot\vec{B} + |\vec{B}|^2$$

Cauchy-Schwarz $\longrightarrow \leq |\vec{A}|^2 + 2|\vec{A}||\vec{B}| + |\vec{B}|$



$$= (|\vec{A}| + |\vec{B}|)^2 .$$

$$\Rightarrow |\vec{A}+\vec{B}| \leq |\vec{A}| + |\vec{B}| .$$

☒

c) <u>Reverse triangle inequality</u>: $\quad \big|\, |\vec{A}| - |\vec{B}| \,\big| \leq |\vec{A}-\vec{B}|$.

absolute value of scalar

magnitude / norm of vector

i.e. we need to show $\quad |\vec{A}| - |\vec{B}| \leq |\vec{A}-\vec{B}|$ <u>and</u> $|\vec{B}| - |\vec{A}| \leq |\vec{A}-\vec{B}|$.

<u>Proof</u>:

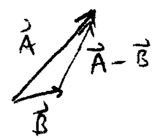
$$\vec{A} = (\vec{A}-\vec{B}) + \vec{B}$$

$$\Rightarrow |\vec{A}| = |(\vec{A}-\vec{B}) + \vec{B}| \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} |\vec{A}-\vec{B}| + |\vec{B}|$$

$$\Rightarrow |\vec{A}| - |\vec{B}| \leq |\vec{A}-\vec{B}| . \qquad ③$$

Similarly, reversing the roles of $\vec{A}$ and $\vec{B}$, we find

$$|\vec{B}| - |\vec{A}| \leq |\vec{B}-\vec{A}| = |\vec{A}-\vec{B}| . \qquad ④$$



Combining ③ and ④,

$$\big|\, |\vec{A}| - |\vec{B}| \,\big| \leq |\vec{A}-\vec{B}|$$

☒

The point of these exercises is that these inequalities depend only on abstract properties of vector spaces and the inner product; thus they will hold also in other mathematical settings satisfying the properties of vector spaces and the inner product.

[See the discussion on the homework sheet.]

2. Linear Independence, the Gram-Schmidt process, Orthogonal Transformations

You should please familiarize yourselves with the Gram-Schmidt process and other material discussed on the homework sheets. In this problem, we verify that alternative approaches to change of basis from one orthonormal coordinate system to another, give the same answer.

[Note: all column vectors are given relative to the canonical Cartesian basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ unless otherwise stated.]

a) $\vec{a}_1 = \hat{\imath} - \hat{\jmath} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\vec{a}_2 = 3\hat{\imath} + \hat{\jmath} - \hat{k} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$, $\vec{a}_3 = 2\hat{\jmath} + \hat{k} = \begin{pmatrix}0\\2\\1\end{pmatrix}$

Check linear independence: show that the system $t_1\vec{a}_1 + t_2\vec{a}_2 + t_3\vec{a}_3 = \vec{0}$ has only the unique solution, or equivalently:

$$[\vec{a}_1, \vec{a}_2, \vec{a}_3] = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \det\begin{pmatrix}1 & -1 & 0\\ 3 & 1 & -1\\ 0 & 2 & 1\end{pmatrix}$$

$$= 1\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix} + 1\begin{vmatrix}3 & -1\\ 0 & 1\end{vmatrix} = 3 + 3 = 6 \neq 0$$

$\Rightarrow$ independent.

Gram-Schmidt process:

$$\vec{b}_1 = \vec{a}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\vec{b}_2 = \vec{a}_2 - \text{proj}_{\vec{b}_1}\vec{a}_2 = \vec{a}_2 - \frac{\vec{a}_2\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1}\vec{b}_1 = \begin{pmatrix}3\\1\\-1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$$

$$\vec{b}_3 = \vec{a}_3 - \text{proj}_{\vec{b}_1}\vec{a}_3 - \text{proj}_{\vec{b}_2}\vec{a}_3 = \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{-2}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} - \frac{3}{9}\begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}1/3\\1/3\\4/3\end{pmatrix}$$

$$\Rightarrow \vec{b}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix},\quad \vec{b}_2 = \begin{pmatrix}2\\2\\-1\end{pmatrix},\quad \vec{b}_3 = \tfrac{1}{3}\begin{pmatrix}1\\1\\4\end{pmatrix}$$

Normalized vectors

$$\hat{e}_1' = \frac{\vec{b}_1}{|\vec{b}_1|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix},\quad \hat{e}_2' = \frac{\vec{b}_2}{|\vec{b}_2|} = \frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix},\quad \hat{e}_3' = \frac{1}{\sqrt{18}}\begin{pmatrix}1\\1\\4\end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix}1\\1\\4\end{pmatrix}$$

Check: $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$

b) Vectors $\vec{A} = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$, $\vec{B} = \hat{\imath} - \hat{k}$ ie $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Write the vectors in the new basis:

$$\vec{A} = A_1' \hat{e}_1' + A_2' \hat{e}_2' + A_3' \hat{e}_3'$$

- an orthonormal set of vectors: find components A_1', A_2', A_3' by projection:

$$\left[\vec{A} \cdot \hat{e}_1' = A_1' \hat{e}_1' \cdot \hat{e}_1' + A_2' \hat{e}_2' \cdot \hat{e}_1' + A_3' \hat{e}_3' \cdot \hat{e}_1 = A_1' + 0 + 0 = A_1' \right]$$

$\Rightarrow \boxed{A_1' = \vec{A} \cdot \hat{e}_1'}$ Similarly for A_2', A_3'.

So $A_1' = \vec{A} \cdot \hat{e}_1' = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot 7 = 7/\sqrt{2}$, $A_2' = \vec{A} \cdot \hat{e}_2' = 0$, $A_3' = \frac{9}{\sqrt{18}} = \frac{3}{\sqrt{2}}$

Similarly $B_1' = \vec{B} \cdot \hat{e}_1' = \frac{1}{\sqrt{2}}$, $B_2' = \frac{3}{3} = 1$, $B_3' = \frac{-3}{\sqrt{18}} = -\frac{1}{\sqrt{2}}$

$$\left[\text{Check: } \vec{A} = A_1' \hat{e}_1' + A_2' \hat{e}_2' + A_3' \hat{e}_3' = \frac{7}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0 \cdot \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \frac{3}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \right.$$

$$\left. = \begin{pmatrix} 7/2 + 0 + \frac{1}{2} \\ -7/2 + 0 + 1/2 \\ 0 + 0 + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \quad \checkmark \right]$$

c) Transformation matrix

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \quad \text{where } J_{ij} = \hat{e}_i \cdot \hat{e}_j'$$

(columns of J are new basis vectors written w.r.t. old basis)

$$\Rightarrow J = \begin{pmatrix} \frac{1}{\sqrt{2}} & 2/3 & \frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2/3 & \frac{1}{3\sqrt{2}} \\ 0 & -1/3 & +\frac{4}{3\sqrt{2}} \end{pmatrix}, \quad J^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 2/3 & 2/3 & -1/3 \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 4/3\sqrt{2} \end{pmatrix}$$

$$J J^T = \begin{pmatrix} 1/\sqrt{2} & 2/3 & \frac{1}{3\sqrt{2}} \\ -1/\sqrt{2} & 2/3 & 1/3\sqrt{2} \\ 0 & -1/3 & 4/3\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 2/3 & 2/3 & -1/3 \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 4/3\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \quad \checkmark$$

d) Components A_i', B_i' using the transformation matrix:

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 2/3 & 2/3 & -1/3 \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 4/3\sqrt{2} \end{pmatrix}}_{J^T} \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7/\sqrt{2} \\ 0 \\ 3/\sqrt{2} \end{pmatrix}$$ as computed in b) above.

Check: $$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 2/3 & \frac{1}{3\sqrt{2}} \\ -1/\sqrt{2} & 2/3 & \frac{1}{3\sqrt{2}} \\ 0 & -1/3 & \frac{4}{3\sqrt{2}} \end{pmatrix}}_{J} \begin{pmatrix} 7/\sqrt{2} \\ 0 \\ 3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$ as expected. ✓

$$\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 0 \\ 2/3 & 2/3 & -1/3 \\ \frac{1}{3\sqrt{2}} & 1/3\sqrt{2} & 4/3\sqrt{2} \end{pmatrix}}_{J^T} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{pmatrix}$$ ✓

Check: $$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 2/3 & \frac{1}{3\sqrt{2}} \\ -1/\sqrt{2} & 2/3 & \frac{1}{3\sqrt{2}} \\ 0 & -1/3 & 4/3\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$ ✓

e) Dot product $\vec{A} \cdot \vec{B}$:

evaluated in old coordinate system

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = 4 + 0 - 2 = 2$$

in new coordinate system

$$\vec{A} \cdot \vec{B} = A_1' B_1' + A_2' B_2' + A_3' B_3' = \tfrac{7}{2} + 0 - \tfrac{3}{2} = 2$$ ✓

as expected.

Comment on Section 1.8 #18

- the lines do not intersect

(contrary to the implication in the question...)