MATH 252-3 **Spring 2004**

Vector Calculus

Final Exam Wednesday, 14 April 2004

Attempt all of the following questions; there are 12 problems, for a total of 120 points. The total time available is three hours (180 minutes).

1. (8 points)
   Consider a particle whose trajectory traces out a space curve $\mathbf{R} = \mathbf{R}(t)$ as a function of time t. Derive the formula
   $$\mathbf{a}(t) = \frac{d^2 s}{dt^2}\mathbf{T} + k\left(\frac{ds}{dt}\right)^2 \mathbf{N}$$
   for the acceleration $\mathbf{a}$, where $\mathbf{T}$ is the unit tangent, $\mathbf{N}$ is the unit principal normal to the curve, s is the distance (arc length) travelled, and k is the curvature.
   Interpret this result.

2. (8 points)
   For which value of β is the vector field
   $$\mathbf{F} = (3 + 2xy^2)\mathbf{i} + (2x^2 y + \beta z)\mathbf{j} + 2y\mathbf{k}$$
   conservative? For this value of β, compute the line integral
   $$\int_0^{\pi/4} \mathbf{F} \cdot d\mathbf{R}$$
   along the helical path
   $$\mathbf{R}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} - 5t\,\mathbf{k}\ .$$

3. (14 points)
   Consider the region V bounded by the cylindrical surface $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$; and let $S = \partial V$ be the closed surface bounding V, with the usual outwards orientation. Define the vector field $\mathbf{F}$ by
   $$\mathbf{F} = -x\,\mathbf{i} + (2z - 1)y\,\mathbf{j} + z^2\,\mathbf{k}\ .$$
   Evaluate
   $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$
   (a) by direct computation of the surface integral;
   (b) using the Divergence Theorem.

4. (8 points)
Let S be the portion of the paraboloid $z = a^2 - x^2 - y^2$ above the x-y plane (with $\mathbf{n}\cdot\mathbf{k} > 0$); and define the vector field

$$\mathbf{F} = 3y\,\mathbf{i} - 2y(1+z^2)\,\mathbf{j} + \arctan x^2\,\mathbf{k}\ .$$

Evaluate the flux of curl $\mathbf{F}$ through the surface S, that is, $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, using one or two applications of Stokes' Theorem.

5. (8 points)
Let V be the region bounded by the cone $z^2 = x^2 + y^2$ and the unit sphere $x^2+y^2+z^2 = 1$; and let S be the surface enclosing V. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = (x + yz^2 e^{-yz})\mathbf{i} + (yz - x^3 e^{xz})\mathbf{j} + (-z + \sin x \cos y)\mathbf{k}\ .$$

Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}\ .$$

6. (10 points)
Let S_t be the square with corners $(0,0,t)$, $(0,1,t)$, $(1,0,t)$ and $(1,1,t)$, with boundary C_t, and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(\mathbf{R},t) = xzt\,\mathbf{k}\ .$$

Evaluate the surface integral

$$\Phi(t) = \iint_{S_t} \mathbf{F}(\mathbf{R},t) \cdot d\mathbf{S}\ ,$$

and verify the flux transport theorem

$$\frac{d\Phi}{dt} = \iint_{S_t} \left(\frac{\partial \mathbf{F}}{\partial t} + (\nabla \cdot \mathbf{F})\,\mathbf{v} \right) \cdot d\mathbf{S} + \oint_{C_t} (\mathbf{F} \times \mathbf{v}) \cdot d\mathbf{R}\ ,$$

where $\mathbf{v}(\mathbf{R},t)$ is the velocity of points of S_t.

7. (8 points)
State Green's Theorem in the Plane.
Prove the theorem for the case of a simply connected, convex region D (in which each vertical and horizontal line intersects the boundary in two points).
Indicate how you would prove Green's Theorem for more general regions in the plane.

8. (10 points)
   Consider the torus (with major radius A, minor radius a, toroidal angle u, poloidal angle v):

$$\begin{aligned} x &= A\cos u + a\cos u\cos v\ , \\ y &= A\sin u + a\sin u\cos v\ , \\ z &= a\sin v \\ &\quad (0 \le u \le 2\pi\ , \quad 0 \le v \le 2\pi\ , \quad 0 < a < A)\ . \end{aligned}$$

   Show that the surface element is

$$\begin{aligned} d\mathbf{S} &= [a\cos u\cos v(A + a\cos v)\,\mathbf{i} + a\sin u\cos v(A + a\cos v)\,\mathbf{j} \\ &\quad + a\sin v(A + a\cos v)\,\mathbf{k}]\,du\,dv \\ &= a(A + a\cos v)\,[\cos u\cos v\,\mathbf{i} + \sin u\cos v\,\mathbf{j} + \sin v\,\mathbf{k}]\ du\,dv\ , \end{aligned}$$

   and thus show that the element of area on the torus is

$$dS = a(A + a\cos v)\,du\,dv\ .$$

   Integrate to obtain the formula for the surface area of the torus: $4\pi^2 Aa$.

9. (16 points)
   Consider the transformation from the Cartesian coordinates (x, y, z) to curvilinear coordinates (u_1, u_2, u_3) given by

$$\begin{aligned} x &= \alpha\, u_1 u_2\ , \\ y &= u_3\ , \\ z &= u_1^2 - u_2^2. \end{aligned}$$

   (a) For which value(s) of the constant α does the transformation describe an *orthogonal* curvilinear coordinate system?
   (b) For which value(s) of α found in (a) is (u_1, u_2, u_3) a *right-handed* coordinate system?
   (c) Compute the scale factors h_1, h_2 and h_3 and unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for any value of α found in (b), and give the element of volume dV in this coordinate system.
   (d) Find the divergence of the vector field $\mathbf{F} = u_3\mathbf{e}_1$.

10. (8 points)
    Let ϕ and ψ be two sufficiently smooth (twice continuously differentiable) scalar fields in a region V with boundary $S = \partial V$.

    (a) By applying the Divergence Theorem to the vector field $\mathbf{F} = \psi\nabla\phi$, prove Green's first formula

$$\iiint_V \psi\nabla^2\phi\, dV = \iint_S \psi\nabla\phi\cdot d\mathbf{S} - \iiint_V \nabla\psi\cdot\nabla\phi\, dV\ .$$

    Hence show that for any sufficiently smooth scalar field ϕ,

$$\iiint_V \phi\nabla^2\phi\, dV = \iint_S \phi\nabla\phi\cdot d\mathbf{S} - \iiint_V |\nabla\phi|^2\, dV\ . \qquad (1)$$

(b) Suppose ϕ is a harmonic function, that is, it satisfies Laplace's equation $\nabla^2\phi = 0$ in V; and also suppose that ϕ vanishes on the boundary, $\phi = 0$ on $S = \partial V$. Use the formula (1) from (a) to show that in this case, the gradient $\nabla\phi$ must vanish identically in V.
Hence deduce that such a function ϕ is constant (and consequently identically zero, from the boundary values $\phi = 0$ on S).

11. (12 points)
Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the usual position vector, and let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a *constant* vector.

(a) Demonstrate the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$$

in *two* ways:

i. using tensor notation; and
ii. by appropriate use of vector identities given on the formula sheet.

(b) Let S be a smooth oriented surface with normal $\mathbf{n}$, and let its boundary be $C = \partial S$ with unit tangent $\mathbf{T}$. Prove that

$$\oint_C (\mathbf{R} \times \mathbf{T})\, ds = 2 \iint_S \mathbf{n}\, dS \ .$$

[Hint: Take the dot product of both sides with a constant vector $\mathbf{A}$.]

12. (10 points)

(a) Faraday's Law of Induction states that

$$\oint_C \mathbf{E} \cdot d\mathbf{R} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} \ ,$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, and the closed curve C is the boundary of the arbitrary surface S in a simply connected region. Deduce that

$$\iint_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} = 0 \ ,$$

and hence obtain the third Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ . \tag{2}$$

(b) Use another of Maxwell's equations (in a simply connected domain) to explain why there exists a vector potential $\mathbf{A}$ so that $\mathbf{B} = \nabla \times \mathbf{A}$.
[Note that here the vector field $\mathbf{A}$ is *not* constant; the use of $\mathbf{A}$ for the magnetic potential is conventional notation.]

(c) Substitute $\mathbf{B} = \nabla \times \mathbf{A}$ into equation (2) (and interchange space and time derivatives) to show that there must exist a scalar potential function ϕ so that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi \ ;$$

that is, we can write the electric field $\mathbf{E}$ in terms of scalar and vector potentials as $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$.

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Vector Calculus

Formula Sheet

- <u>*Vector Identities*</u>

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = [\mathbf{A},\mathbf{C},\mathbf{D}]\mathbf{B} - [\mathbf{B},\mathbf{C},\mathbf{D}]\mathbf{A} \qquad [\mathbf{A},\mathbf{B},\mathbf{C}] = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$$

$$\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1 \qquad \nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\phi$$

$$\nabla\times(\phi\mathbf{F}) = \phi\nabla\times\mathbf{F} + \nabla\phi\times\mathbf{F} \qquad \nabla f(\phi) = \frac{df}{d\phi}\nabla\phi$$

$$\nabla\cdot\mathbf{R} = 3 \qquad \nabla\times\mathbf{R} = \mathbf{0}$$

$$\mathbf{F}\cdot\nabla\mathbf{R} = \mathbf{F} \qquad \nabla(\mathbf{A}\cdot\mathbf{R}) = \mathbf{A}$$

$$\begin{aligned}
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}) \\
\nabla\times(\mathbf{F}\times\mathbf{G}) &= (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G} \\
\nabla(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) \\
\nabla\times(\nabla\times\mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}
\end{aligned}$$

Vector potential: $\mathbf{G}(\mathbf{R}) = \int_0^1 t\mathbf{F}\times\frac{d\mathbf{r}}{dt}dt$, $\quad \mathbf{r}(t) = \mathbf{R}_0 + t(\mathbf{R}-\mathbf{R}_0)$

- *Frenet Formulas*

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -k\mathbf{T} + \tau\mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

- *Flux Transport Theorem*

$$\frac{d\Phi}{dt} = \frac{d}{dt}\iint_{S_t}\mathbf{F}(\mathbf{R},t)\cdot d\mathbf{S} = \iint_{S_t}\left(\frac{\partial\mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\,\mathbf{v}\right)\cdot d\mathbf{S} + \oint_{C_t}\mathbf{F}\times\mathbf{v}\cdot d\mathbf{R}$$

- *Reynolds Transport Theorem*

$$\frac{d}{dt}\iiint_{V_t} f(\mathbf{R},t)\,dV = \iiint_{V_t}\left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{v})\right)dV = \iiint_{V_t}\left(\frac{Df}{Dt} + f\nabla\cdot\mathbf{v}\right)dV$$

- <u>*General Orthogonal Curvilinear Coordinates*</u>

Displacement vector: $d\mathbf{R} = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$

Arc length: $ds = \left(h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2\right)^{1/2}$

Volume element: $dV = h_1h_2h_3\,du_1\,du_2\,du_3$

Gradient: $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3$

Divergence: $\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_3h_1) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right]$

Curl: $\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}$

Laplacian: $\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$

- *Cylindrical Coordinates*

$$\text{Definitions:} \quad x = \rho\cos\theta\ , \quad y = \rho\sin\theta\ , \quad z = z$$

$$\text{Displacement vector:} \quad d\mathbf{R} = d\rho\,\mathbf{e}_\rho + \rho\,d\theta\,\mathbf{e}_\theta + dz\,\mathbf{e}_z$$

$$\text{Arc length:} \quad ds = \left(d\rho^2 + \rho^2\,d\theta^2 + dz^2\right)^{1/2}$$

$$\text{Volume element:} \quad dV = \rho\,d\rho\,d\theta\,dz$$

$$\text{Gradient:} \quad \nabla f = \frac{\partial f}{\partial\rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z$$

$$\text{Divergence:} \quad \nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\text{Curl:} \quad \nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

$$\text{Laplacian:} \quad \nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

- *Spherical Coordinates*

$$\text{Definitions:} \quad x = r\sin\phi\cos\theta\ , \quad y = r\sin\phi\sin\theta\ , \quad z = r\cos\phi$$

$$\text{Displacement vector:} \quad d\mathbf{R} = dr\,\mathbf{e}_r + r\,d\phi\,\mathbf{e}_\phi + r\sin\phi\,d\theta\,\mathbf{e}_\theta$$

$$\text{Arc length:} \quad ds = \left(dr^2 + r^2\,d\phi^2 + r^2\,\sin^2\phi\,d\theta^2\right)^{1/2}$$

$$\text{Volume element:} \quad dV = r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$\text{Gradient:} \quad \nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\phi}\,\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial\theta}\,\mathbf{e}_\theta$$

$$\text{Divergence:} \quad \nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(F_\phi\sin\phi\right) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}$$

$$\text{Curl:} \quad \nabla\times\mathbf{F} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & (r\sin\phi)\mathbf{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial\theta} \\ F_r & rF_\phi & (r\sin\phi)F_\theta \end{vmatrix}$$

$$\text{Laplacian:} \quad \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

- *Maxwell's Equations*

Maxwell's equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in free space, in the absence of magnetic or polarizable media, in SI (mks) units; with charge density ρ, current density $\mathbf{J}$, and universal constants ϵ_0 (permittivity of free space) and μ_0 (permeability of free space) (where $\epsilon_0\mu_0 = c^{-2}$):

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Also: Gauss' Law: $\displaystyle\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$ (Q : total enclosed charge)