MATH 252-3 **Spring 2005**

Vector Calculus

Formula Sheet for Final Exam — Final Exam Tuesday 19 April 2005

[This formula sheet will be given on the final exam; you may find it useful elsewhere as well. No calculators or notes will be permitted on the exam otherwise.]

- *Fourier Coefficients*

  Fourier polynomial of degree n of $f(x)$ on $[-\pi,\pi]$: $F_n(x) = a_0 + \sum_{k=1}^{n} [a_k \cos kx + b_k \sin kx]$ where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\ , \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\ , \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\ .$$

- *Vector Identities*

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = [\mathbf{A},\mathbf{C},\mathbf{D}]\mathbf{B} - [\mathbf{B},\mathbf{C},\mathbf{D}]\mathbf{A} \qquad [\mathbf{A},\mathbf{B},\mathbf{C}] = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$$

$$\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1 \qquad \nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\phi$$

$$\nabla\times(\phi\mathbf{F}) = \phi\nabla\times\mathbf{F} + \nabla\phi\times\mathbf{F} \qquad \nabla f(\phi) = \frac{df}{d\phi}\nabla\phi$$

$$\nabla\cdot\mathbf{R} = 3 \qquad \nabla\times\mathbf{R} = \mathbf{0}$$

$$\mathbf{F}\cdot\nabla\mathbf{R} = \mathbf{F} \qquad \nabla(\mathbf{A}\cdot\mathbf{R}) = \mathbf{A}$$

$$\begin{aligned}
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})\\
\nabla\times(\mathbf{F}\times\mathbf{G}) &= (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G}\\
\nabla(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})\\
\nabla\times(\nabla\times\mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}
\end{aligned}$$

  Vector potential: $\mathbf{G}(\mathbf{R}) = \int_0^1 t\mathbf{F}\times\frac{d\mathbf{r}}{dt}dt\ , \qquad \mathbf{r}(t) = \mathbf{R}_0 + t(\mathbf{R}-\mathbf{R}_0)$

- *Frenet Formulas* $\quad \frac{d\mathbf{T}}{ds} = k\mathbf{N}\ , \qquad \frac{d\mathbf{N}}{ds} = -k\mathbf{T} + \tau\mathbf{B}\ , \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$

- *General Orthogonal Curvilinear Coordinates*

  Displacement vector: $d\mathbf{R} = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$

  Arc length: $ds = \left(h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2\right)^{1/2}$

  Volume element: $dV = h_1h_2h_3\,du_1\,du_2\,du_3$

  Gradient: $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3$

  Divergence: $\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_3h_1) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right]$

  Curl: $\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3}\\ F_1h_1 & F_2h_2 & F_3h_3\end{vmatrix}$

  Laplacian: $\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$

- *Cylindrical Coordinates*

$$\text{Definitions:} \quad x = \rho\cos\theta\ , \quad y = \rho\sin\theta\ , \quad z = z$$

$$\text{Displacement vector:} \quad d\mathbf{R} = d\rho\,\mathbf{e}_\rho + \rho\,d\theta\,\mathbf{e}_\theta + dz\,\mathbf{e}_z$$

$$\text{Arc length:} \quad ds = \left(d\rho^2 + \rho^2\,d\theta^2 + dz^2\right)^{1/2}$$

$$\text{Volume element:} \quad dV = \rho\,d\rho\,d\theta\,dz$$

$$\text{Gradient:} \quad \nabla f = \frac{\partial f}{\partial \rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z$$

$$\text{Divergence:} \quad \nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\text{Curl:} \quad \nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

$$\text{Laplacian:} \quad \nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

- *Spherical Coordinates*

$$\text{Definitions:} \quad x = r\sin\phi\cos\theta\ , \quad y = r\sin\phi\sin\theta\ , \quad z = r\cos\phi$$

$$\text{Displacement vector:} \quad d\mathbf{R} = dr\,\mathbf{e}_r + r\,d\phi\,\mathbf{e}_\phi + r\sin\phi\,d\theta\,\mathbf{e}_\theta$$

$$\text{Arc length:} \quad ds = \left(dr^2 + r^2\,d\phi^2 + r^2\ \sin^2\phi\,d\theta^2\right)^{1/2}$$

$$\text{Volume element:} \quad dV = r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$\text{Gradient:} \quad \nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\phi}\,\mathbf{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial\theta}\,\mathbf{e}_\theta$$

$$\text{Divergence:} \quad \nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(F_\phi\sin\phi\right) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}$$

$$\text{Curl:} \quad \nabla\times\mathbf{F} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & (r\sin\phi)\mathbf{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial\theta} \\ F_r & rF_\phi & (r\sin\phi)F_\theta \end{vmatrix}$$

$$\text{Laplacian:} \quad \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

- *Maxwell's Equations*

Maxwell's equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in free space, in the absence of magnetic or polarizable media, in SI (mks) units; with charge density ρ, current density $\mathbf{J}$, and universal constants ϵ_0 (permittivity of free space) and μ_0 (permeability of free space) (where $\epsilon_0\mu_0 = c^{-2}$):

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Also: Gauss' Law: $\displaystyle\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$ (Q : total enclosed charge)