MATH 252-3 **Spring 2004**

Vector Calculus

Midterm Exam Friday, 20 February 2004

Complete all of the following questions; there are 5 problems, for a total of 50 points. You should aim to spend about a minute per point, for example 10 minutes for a 10 point problem.

1. (10 points)
   Use tensor notation to prove the following identities:
   (a) $\nabla \cdot (f\mathbf{G}) = f\nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla f$,
   where f is a differentiable scalar field and $\mathbf{G}$ is a differentiable vector field.
   (b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$
   (c) Use the result in (b) to show the vector identity
   $$\mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|^2}\mathbf{A} + \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{|\mathbf{A}|^2}\ , \qquad (\mathbf{A} \neq \mathbf{0})$$
   and interpret the result geometrically.

2. (10 points)
   For the vector field $\mathbf{F}$ given in Cartesian coordinates as
   $$\mathbf{F} = (x^2y + xz)\mathbf{i} + xy\mathbf{j} + e^z\mathbf{k}\ ,$$
   compute
   (a) $\nabla \cdot \mathbf{F}$ and $\nabla(\nabla \cdot \mathbf{F})$
   (b) $\nabla^2\mathbf{F}$
   (c) $\nabla \times \mathbf{F}$ and $\nabla \times (\nabla \times \mathbf{F})$
   (d) Use your results from (a), (b) and (c) to verify that $\mathbf{F}$ satisfies the vector identity
   $$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$

3. (15 points)
   A particle travels in a helical path, with its position vector given by
   $$\mathbf{R}(t) = 4\cos t\,\mathbf{i} + 3t\,\mathbf{j} + 4\sin t\,\mathbf{k}, \qquad t \geq 0.$$
   (a) Find the velocity, speed and acceleration of the particle.
   (b) Parametrize the path in terms of the arc length along the path.
   (c) Compute the curvature of the path, and the tangent, principal normal and binormal vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ at the initial time $t = 0$.
   (d) Consider the change of coordinates from the standard Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ at $t = 0$. Find the transformation (Jacobian) matrix for this coordinate change.

4. (10 points)

   (a) Show that any position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ can be expressed in terms of the Frenet frame as

   $$\mathbf{R} = (\mathbf{R}\cdot\mathbf{T})\mathbf{T} + (\mathbf{R}\cdot\mathbf{N})\mathbf{N} + (\mathbf{R}\cdot\mathbf{B})\mathbf{B}.$$

   (b) If the curve $\mathbf{R}(t)$ lies on the sphere $|\mathbf{R}(t)| = \textit{constant}$, show that

   $$\mathbf{R} = -\rho\mathbf{N} - \frac{1}{\tau}\frac{d\rho}{ds}\mathbf{B}\ ,$$

   where ρ is the radius of curvature and τ is the torsion.
   [Hint: *repeatedly* differentiate $\mathbf{R}\cdot\mathbf{R} = \textit{constant}$ with respect to arc length s, and use the Frenet formulas

   $$\frac{d\mathbf{T}}{ds} = k\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -k\mathbf{T} + \tau\mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}. \qquad]$$

5. (5 points)
   Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three *unit* vectors satisfying $[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \neq 0$, and let f be a scalar field, differentiable at a point P.
   Show that if one knows the values of the directional derivatives in the directions of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, that is, $D_{\mathbf{a}}f$, $D_{\mathbf{b}}f$ and $D_{\mathbf{c}}f$, then one can compute the directional derivatives in every direction.