Math 252 Final Exam Solutions

Question 1(a)

$$\iint_{S_{whole}} \vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV = 0 \quad \text{because } \vec{\nabla}\cdot\vec{F} = 0$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} + \iint_{bottom} \vec{F}\cdot d\vec{S}' = 0$$

on the bottom, $\hat{n}' = -\hat{k}$ $\therefore \vec{F}\cdot\hat{n}' = -2$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} - 2\underbrace{\iint_{bottom} dS}_{2\cdot 4\pi m^2} = 0$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = 8\pi m^2$$

Question 1(b)

$\vec{F} = \vec{\nabla}\times\vec{G}$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{\nabla}\times\vec{G}\cdot dS = \int_C \vec{G}\cdot d\vec{R} = \iint_{bottom} \vec{\nabla}\times\vec{G}\cdot dS$$

$$= \iint_{bottom} \vec{F}\cdot d\vec{S}$$

In this case $\hat{n} = \hat{k}$ $\therefore \vec{F}\cdot\hat{n} = 2$

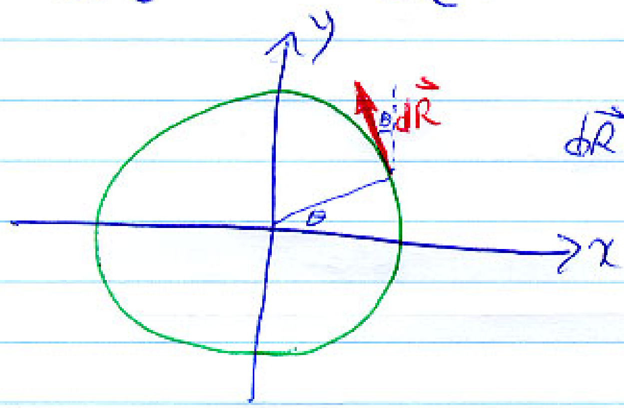
$$\therefore \iint_S \vec{F}\cdot d\vec{S} = 2\iint_{bottom} dS = 8\pi m^2$$

Question 2

$$\iint_M (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{R}$$

where C is a circle in the xy-plane with radius 1
Note that $z=0$ in the x-y plane, and also that $d\vec{R}$ is in the xy plane so we can ignore the $\hat{k}$ component of $\vec{F}$

$$\therefore \int_C \vec{F} \cdot d\vec{R} = \int_C (y\hat{i} - x\hat{j}) \cdot d\vec{R}$$



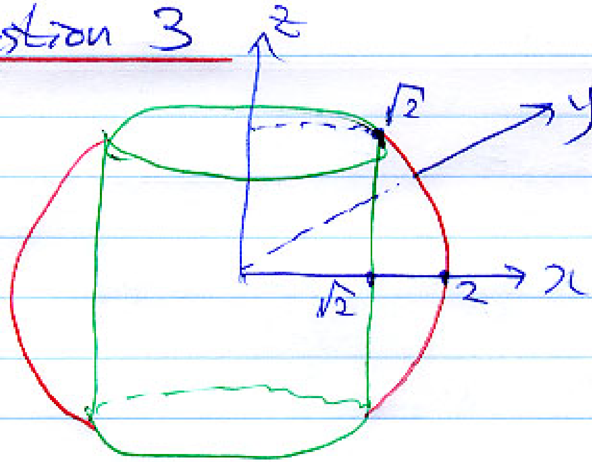
$$d\vec{R} = d\theta \left[\cos\theta\, \hat{j} - \sin\theta\, \hat{i} \right]$$

$$x = \cos\theta$$
$$y = \sin\theta$$

$$\therefore \int_C \vec{F} \cdot d\vec{R} = \int_0^{2\pi} (\sin\theta\, \hat{i} - \cos\theta\, \hat{j}) \cdot (\cos\theta\, \hat{j} - \sin\theta\, \hat{i})\, d\theta$$

$$= \int_0^{2\pi} (-\sin^2\theta - \cos^2\theta)\, d\theta = -\int_0^{2\pi} d\theta = \underline{-2\pi}$$

Question 3



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV = 3\iiint_V dV$$

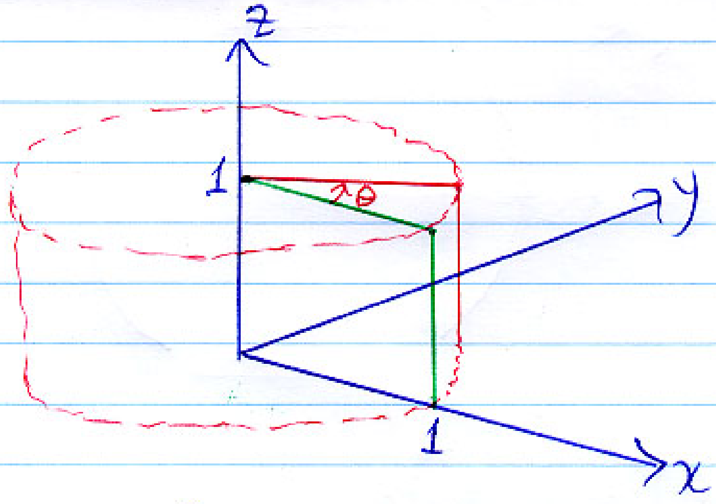
$$= 3\int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-\rho^2}} dz\, d\rho\, 2\pi\rho = 12\pi\int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-\rho^2}} \rho\, dz\, d\rho$$

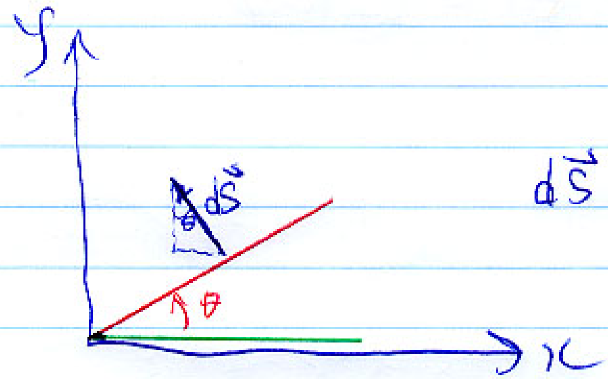
$$= 12\pi\int_{\sqrt{2}}^{2} \rho\sqrt{4-\rho^2}\, d\rho$$

$$= 12\pi(-)\frac{2}{3\cdot 2}(4-\rho^2)^{3/2}\Big|_{\sqrt{2}}^{2} = -4\pi\,(4-\rho^2)^{3/2}\Big|_{\sqrt{2}}^{2}$$

$$= -4\pi\left[0-2^{3/2}\right] = 4\pi\cdot 2^{3/2} = \underline{8\pi\sqrt{2}}$$

Question 4





$d\vec{S} = |dS| \left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

and $|dS| = d\rho\, dz$

$\therefore d\vec{S} = d\rho\, dz \left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

but $\theta = \omega t$

$\therefore d\vec{S} = \left[-\sin(\omega t)\,\hat{i} + \cos(\omega t)\,\hat{j}\right] d\rho\, dz$

And $\vec{F} = -\sin(\omega t)\,\hat{i} + \cos(\omega t)\,\hat{j}$

$\phi(t) = \iint_{S_t} \vec{F} \cdot d\vec{S} = \iint_{S_t} \left[\sin^2(\omega t) + \cos^2(\omega t)\right] d\rho\, dz$

$= \iint_{S_t} d\rho\, dz = 1 \qquad \therefore \dfrac{d\phi(t)}{dt} = 0$

$\therefore$ LHS of Flux theorem is zero. What about RHS?

$|\vec{v}| = \rho\omega$ in the same direction as $d\vec{S}$

$\therefore \vec{v} = \rho\omega\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

$\vec{v} = \rho\omega\left(-\sin(\omega t)\,\hat{i} + \cos(\omega t)\,\hat{j}\right)$

$\therefore \vec{F} \parallel \vec{v}$ so $\vec{F}\times\vec{v} = 0$

$\therefore \oint_{C_t} (\vec{F}\times\vec{v})\cdot d\vec{R} = 0$

Now $\dfrac{\partial \vec{F}}{\partial t} = -\omega\cos(\omega t)\,\hat{i} - \omega\sin(\omega t)\,\hat{j}$

so $\dfrac{\partial \vec{F}}{\partial t}\cdot d\vec{S} = -\omega\left[\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\right]\cdot$

$\cdot\left[-\sin(\omega t)\,\hat{i} + \cos(\omega t)\,\hat{j}\right] d\rho\, dz$

$= -\omega\left[-\cos(\omega t)\sin(\omega t) + \cos(\omega t)\sin(\omega t)\right] d\rho\, dz = 0$

$\therefore \iint_{S_t} \dfrac{\partial \vec{F}}{\partial t}\cdot d\vec{S} = 0$

Also $\vec{F} = -\sin(\omega t)\,\hat{i} + \cos(\omega t)\,\hat{j}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$= -\dfrac{y}{\sqrt{x^2+y^2}}\,\hat{i} + \dfrac{x}{\sqrt{x^2+y^2}}\,\hat{j}$

$\therefore \vec{\nabla}\cdot\vec{F} = \dfrac{+yx}{(x^2+y^2)^{3/2}} - \dfrac{xy}{(x^2+y^2)^{3/2}} = 0$

$\therefore \int_{S_t} (\vec{\nabla}\cdot\vec{F})\,\vec{v}\cdot d\vec{S} = 0$

$\therefore$ RHS of Flux Theorem $= 0$ Also

Question 5

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\iiint_V \vec{\nabla}\cdot\vec{E}\,dV = \frac{1}{\varepsilon_0}\iiint_V \rho\,dV = \frac{q}{\varepsilon_0}$$

By divergence theorem $\iiint_V \vec{\nabla}\cdot\vec{E}\,dV = \iint_S \vec{E}\cdot d\vec{S}$

$$\therefore \frac{q}{\varepsilon_0} = \iint_S \vec{E}\cdot d\vec{S} \quad \Big\} \text{ Gauss' Law}$$

Question 6

$$\phi(\vec{R}) = -\frac{1}{4\pi}\iiint_{D'} \frac{\rho(\vec{R}')}{|\vec{R}-\vec{R}'|}\,dV'$$

$$\therefore \vec{\nabla}^2\phi(\vec{R}) = -\frac{1}{4\pi}\iiint_{D'} \rho(\vec{R}')\,\delta(\vec{R}-\vec{R}')\,dV'\cdot(-4\pi)$$

$$= \rho(\vec{R})$$

$\therefore \vec{\nabla}^2\phi(\vec{R}) = \rho(\vec{R})$: ~~Laplac~~ Poisson's equation

Question 7

$$\vec{\nabla}\cdot\vec{F} = \lim_{V\to 0}\frac{1}{V}\iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$= \lim_{V\to 0}\frac{1}{V}\iint_S \vec{F}\cdot dS$$

$\therefore$ Divergence of $\vec{F}$ is the net outflax per unit volume, in the infinitesimal limit of small volume.

Question 8

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \lim_{A\to 0}\frac{1}{A}\iint_A(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$$

$$= \lim_{A\to 0}\frac{1}{A}\int_C \vec{F}\cdot d\vec{R}$$ where C encloses S.

$\therefore$ The normal component of curl is the "swirl" around around a surface per unit area, in the infinitesimal limit of small area.

Question 9

$$\vec{F} = \frac{2x}{x^2+y^2}\hat{i} + \frac{2y}{x^2+y^2}\hat{j} + 2z\hat{k}$$

Note: domain not simply connected. Can't use curl test.

Let's try to find a ϕ such that $\vec{F} = \vec{\nabla}\phi$.

$$\frac{\partial\phi}{\partial x} = \frac{2x}{x^2+y^2} \quad \therefore \phi = \int\frac{2x\,dx}{(x^2+y^2)} = \ln(x^2+y^2) + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2} + \frac{\partial f(y,z)}{\partial y} = \frac{2y}{x^2+y^2} \quad \therefore f(y,z) = g(z)$$

$$\therefore \phi = \ln(x^2+y^2) + g(z)$$

$$\frac{\partial \phi}{\partial z} = g'(z) = 2z \qquad \therefore g(z) = z^2 + C$$

$$\therefore \boxed{\phi = \ln(x^2+y^2) + z^2 + C}$$

$\therefore \vec{F}$ is conservative

Question 10

$$x = u_1^2 - u_2^2$$
$$y = 2u_1u_2$$
$$z = u_3$$
$$\therefore \vec{R} = (u_1^2 - u_2^2)\hat{i} + 2u_1u_2\hat{j} + u_3\hat{k}$$

$$\therefore \frac{\partial \vec{R}}{\partial u_1} = 2u_1\hat{i} + 2u_2\hat{j}$$

$$h_1 = \left|\frac{\partial \vec{R}}{\partial u_1}\right| = 2\sqrt{u_1^2+u_2^2}$$

$$\hat{e}_1 = \frac{u_1\hat{i} + u_2\hat{j}}{\sqrt{u_1^2+u_2^2}}$$

$$\cancel{h_2 = \left|\frac{\partial \vec{R}}{\partial u_2}\right|} \qquad \frac{\partial \vec{R}}{\partial u_2} = -2u_2\hat{i} + 2u_1\hat{j}$$

$$\therefore h_2 = \left|\frac{\partial \vec{R}}{\partial u_2}\right| = 2\sqrt{u_1^2+u_2^2}$$

$$\hat{e}_2 = \frac{-u_2\hat{i} + u_1\hat{j}}{\sqrt{u_1^2+u_2^2}}$$

$$\frac{\partial \vec{R}}{\partial u_3} = \hat{k}$$

$$h_3 = \left|\frac{\partial \vec{R}}{\partial u_3}\right| = 1 \qquad \therefore \hat{e}_3 = \hat{k}$$

$\hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = 0$ obviously.

Also $\hat{e}_1 \cdot \hat{e}_2 = \dfrac{1}{u_1^2+u_2^2}(-u_1u_2+u_1u_2) = 0$

$\therefore \hat{e}_1, \hat{e}_2,$ and $\hat{e}_3$ are mutually orthogonal.

$$\hat{e}_1 \times \hat{e}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ -u_2 & u_1 & 0 \end{vmatrix} \frac{1}{(u_1^2+u_2^2)}$$

$$= \left[\hat{i}\cdot 0 - \hat{j}\cdot 0 + \hat{k}(u_1^2+u_2^2)\right]\frac{1}{u_1^2+u_2^2} = \hat{k} = \hat{e}_3$$

$\therefore \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \qquad \therefore$ Right Handed

Question 11

$$\vec{R} = u^2\hat{i} + \sqrt{2}\,uv\,\hat{j} + v^2\hat{k}$$

$$\frac{\partial \vec{R}}{\partial u} = 2u\hat{i} + \sqrt{2}\,v\,\hat{j}$$

$$\frac{\partial \vec{R}}{\partial v} = \sqrt{2}\,u\,\hat{j} + 2v\hat{k}$$

$$d\vec{S} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv$$

$$\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & \sqrt{2}v & 0 \\ 0 & \sqrt{2}u & 2v \end{vmatrix}$$

$$= \hat{i}\, 2\sqrt{2}v^2 - \hat{j}\, 4uv + \hat{k}\, 2\sqrt{2}u^2$$

$$\therefore d\vec{S} = \left(2\sqrt{2}v^2\hat{i} - 4uv\hat{j} + 2\sqrt{2}u^2\hat{k}\right) du\,dv$$

Question 12

(a)

False : The scale factor is the rate at which arc length increases on the i^{th} coordinate curve, with respect to coordinate u_i

(b) ~~Yes~~ True ! A closed curve can be shrunk to a point without getting hung-up on anything.

(c) True

(d) False The domain D must be simply-connected

(e) False This is an example from the book.

(f) False If a field $\vec{F}$ has straight flow lines, then the flow lines of the vector potential $\vec{G}$ curl around those of $\vec{F}$. Not vice-versa!

~~(g)~~ no (g)

(h) yes $\iint_S \vec{R}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{R}\, dV = 3\iiint_V dV$

TRUE

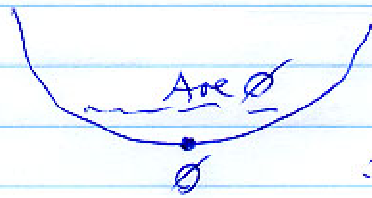
$= 3V$

$\therefore V = \frac{1}{3}\iint_S \vec{R}\cdot d\vec{S}$

(i) FALSE The volume element is $r^2 \sin\phi \, d\phi \, d\theta \, dr$

(j) TRUE

$\phi'' > 0$ is like concave up



$\therefore \phi < \langle\phi\rangle_{ave}$

(k) TRUE : This is the Fundamental Theorem of Vector Analysis

(l) TRUE