MATH 252-3 **Spring 2005**

Vector Calculus

Homework Set 11 (last set!) Due by the Final, Tuesday, 19 April 2005

Course Web Site: `http://www.math.sfu.ca/~ralfw/math252/`

1. *Linear Independence, the Gram-Schmidt Process, and Orthogonal Transformations*
   A set of nonzero vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is *linearly independent* if no nontrivial linear combination of the vectors vanishes; that is, if the vectorial equation

   $$\sum_{i=1}^{n} c_i \mathbf{a}_i = \mathbf{0}$$

   has the unique solution $c_1 = c_2 = \cdots = c_n = 0$. Equivalently, none of the vectors can be expressed as a linear combination of the others; for instance, there are no solutions of $\mathbf{a}_n = c_1' \mathbf{a}_1 + \cdots + c_{n-1}' \mathbf{a}_{n-1}$. For vectors in $\mathbb{R}^3$, linear independence of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ means that the three vectors are not coplanar, that is, that $\mathbf{a}_3$ does not lie in the plane defined by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (assumed non-parallel).

   Given a set of linearly independent vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, we may successively construct a set of mutually orthogonal vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ (spanning the same subspace) using the *Gram-Schmidt process*. The idea is as follows:

   - Choose $\mathbf{b}_1 = \mathbf{a}_1$ (or any nonzero multiple of $\mathbf{a}_1$).
   - Next, we wish to choose $\mathbf{b}_2$ as a vector in the subspace spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, but orthogonal to $\mathbf{b}_1$ (consequently, $\text{span}\{\mathbf{b}_1, \mathbf{b}_2\} = \text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$). Thus define

     $$\mathbf{b}_2 = \mathbf{a}_2 - \lambda \mathbf{b}_1 \ ,$$

     and find λ using the condition $\mathbf{b}_2 \cdot \mathbf{b}_1 = 0$; this gives $\lambda = \mathbf{a}_2 \cdot \mathbf{b}_1 / \mathbf{b}_1 \cdot \mathbf{b}_1$, so that

     $$\mathbf{b}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1} \mathbf{b}_1 = \mathbf{a}_2 - \text{proj}_{\mathbf{b}_1} \mathbf{a}_2.$$

     Note that $\mathbf{b}_2 \neq \mathbf{0}$, since if it were zero that would imply that $\mathbf{a}_2$ and $\mathbf{b}_1 = \mathbf{a}_1$ were linearly dependent, which they are not.
   - Now we can proceed analogously to obtain the remaining orthogonal vectors: At the kth step in the process ($k \geq 2$), we obtain $\mathbf{b}_k$ by taking $\mathbf{a}_k$ and subtracting from it its projection onto the subspace spanned by the previously constructed vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}$:

     $$\mathbf{b}_k = \mathbf{a}_k - \frac{\mathbf{a}_k \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1} \mathbf{b}_1 - \cdots - \frac{\mathbf{a}_k \cdot \mathbf{b}_{k-1}}{\mathbf{b}_{k-1} \cdot \mathbf{b}_{k-1}} \mathbf{b}_{k-1} = \mathbf{a}_k - \sum_{i=1}^{k-1} \frac{\mathbf{a}_k \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i} \mathbf{b}_i \ .$$

     You should verify that this indeed implies that $\mathbf{b}_k \cdot \mathbf{b}_i = 0$ for $i = 1, \ldots, k-1$, and that $\mathbf{b}_k \neq \mathbf{0}$ since $\mathbf{a}_k \notin \text{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}\} = \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}$.
     Once the orthogonal set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ has been constructed recursively, it is straightforward to obtain an *orthonormal* set of vectors $\{\mathbf{e}_1', \ldots, \mathbf{e}_n'\}$ spanning the same subspace, by defining $\mathbf{e}_k' = \mathbf{b}_k / |\mathbf{b}_k|$, $k = 1, \ldots, n$.

   (. . . questions on next page . . .)

In this problem (which you should study and understand, but **don't need to hand in**), we will use the Gram-Schmidt process to construct an orthonormal basis of $\mathbb{R}^3$ from a set of three independent vectors, and then compute the coordinates of given vectors **A** and **B** with respect to the new basis:

Consider the vectors in $\mathbb{R}^3$, expressed in terms of the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{a}_1 = \mathbf{i} - \mathbf{j}\ , \quad \mathbf{a}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}\ , \quad \mathbf{a}_3 = 2\mathbf{j} + \mathbf{k}.$$

Also define the vectors **A** and **B** in this basis as

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \quad \mathbf{B} = \mathbf{i} - \mathbf{k},$$

where the components may be found as $A_i = \mathbf{A}\cdot\mathbf{e}_i$, $B_i = \mathbf{B}\cdot\mathbf{e}_i$, $i = 1, \ldots, 3$. (For simplicity, you may wish to represent the vectors as column vectors containing the components, provided you are clear which basis is being used...)

(a) Show that the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent. Use the Gram-Schmidt process to obtain a set of three mutually orthogonal vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ from $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, where $\mathbf{b}_1 = \mathbf{a}_1$. Also normalize the vectors $\mathbf{b}_i$ to obtain a new orthonormal basis $\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\} = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ for $\mathbb{R}^3$, where $\mathbf{e}'_i$ is a unit vector in the direction of $\mathbf{b}_i$, $i = 1, \ldots, 3$.

(b) Use orthogonal projections onto the new basis vectors to find the components A'_i, B'_i of **A** and **B** with respect to the new basis, where

$$\mathbf{A} = \sum_{i=1}^{3} A'_i \mathbf{e}'_i$$

and similarly for **B**.

(c) Lastly, compute $\mathbf{A}\cdot\mathbf{B}$ in both the old and the new coordinate system, and verify that your answers are the same (that is, $\mathbf{A}\cdot\mathbf{B}$ is a scalar, a quantity independent of the coordinate system).

2. **To hand in**:
In class we applied the Gram-Schmidt process to $\{1, x, x^2, x^3\}$, with the inner product $(f, g) = \int_{-1}^{1} f(x)g(x)\,dx$, to find the first four Legendre polynomials (unscaled),

$$p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2 - \frac{1}{3},\ p_3(x) = x^3 - \frac{3}{5}x\ .$$

(a) Continue the process one more step to derive $p_4(x)$, a degree 4 polynomial orthogonal to $p_j(x)$, $j < 4$, and normalize your answer to find the Legendre polynomial $P_4(x)$ of degree 4, where $P_4(1) = 1$.

(b) Find an approximation $F_4(x)$ to the function $f(x) = x^5$ over the interval $[-1, 1]$ using polynomials of degree ≤ 4, by projecting onto the Legendre basis. Use Maple to verify that you have computed $P_4(x)$ correctly, and to plot $F_4(x)$ and $f(x) = x^5$ on the same axes.

3. **Maple bonus question**:
Find a Taylor polynomial and a Legendre polynomial approximations of degree 2 of the function $f(x) = \cos(\pi x) - 3x$, and plot both on the same set of axes with $f(x)$. Briefly discuss which approximation is more accurate near $x = 0$, and over the entire domain. Also plot the approximations of degree 4.

4. **To hand in**:
Define an inner product for functions $f(x)$, $g(x)$ defined on $[0, \infty)$ by

$$(f, g) = \int_0^\infty f(x)g(x)e^{-x}\, dx,$$

that is, using a weight function $w(x) = e^{-x}$.

Apply the Gram-Schmidt process to the polynomials $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$ to obtain polynomials $L_0(x)$, $L_1(x)$ and $L_2(x)$ which are orthogonal with respect to the above inner product. Check that $L_1(x)$ and $L_2(x)$ are orthogonal. If $q(x)$ is a quadratic polynomial, and

$$q(x) = c_0L_0(x) + c_1L_1(x) + c_2L_2(x),$$

write formulas for the coefficients c_0, c_1 and c_2 (use orthogonality!).
[Answer: These are the *Laguerre polynomials*: $L_0(x) = 1$, $L_1(x) = x - 1$, $L_2(x) = x^2 - 4x + 2$. They form a basis for the set of all (quadratic) polynomials on $[0, \infty)$; coefficients are $c_j = \int_0^\infty q(x)L_j(x)e^{-x}\, dx / \int_0^\infty L_j(x)^2e^{-x}\, dx$.]

5. **For study, not to hand in**:
Verify the fundamental orthogonality relationships for trigonometric functions on $[-\pi, \pi]$ (where m and n are positive integers):

   (a) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \pi\delta_{mn}$ if m and $n > 0$,
   where δ_{mn} is the Kronecker delta ($\delta_{mn} = 1$ if $m = n$, 0 if $m \neq n$).

   (b) $\displaystyle\int_{-\pi}^{\pi} \cos 0x \cos nx\, dx = \int_{-\pi}^{\pi} \cos nx\, dx = 2\pi\delta_{0n}$

   (c) $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0$

6. **Hand in (a,b)**:

   (a) Compute the Fourier coefficients a_0, a_k and b_k, and hence write down the Fourier series, for the function $f(x) = x$ on $[-\pi, \pi]$.

   (b) Write down the Fourier approximation $F_3(x)$ of $f(x) = x$ up to the third harmonic term; use Maple to graph this approximation together with the original function; also graph the 9th harmonic approximation $F_9(x)$.

   Parts (c,d) are optional, and not for the exam:

   (c) Compute the total energy $E = (1/\pi)\int_{-\pi}^{\pi} f(x)^2\, dx$. What fraction of the energy is contained in the constant term and first three harmonics?

   (d) The Fourier series you should have obtained in (a) is

$$x = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots)$$

   (it turns out that the series converges to $f(x) = x$ at each point x where the 2π-periodic extension of f is continuous, that is, on the open interval $-\pi < x < \pi$). Substitute $x = \pi/2$ into both sides of the above formula to demonstrate the following interesting sum of an infinite series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

7. **For study, not to hand in**:
Repeat the calculations of question 6 for the functions:

(a) Square wave: $f(x) = \operatorname{sgn}(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$
(we can define $f(x) = 0$ for $x = 0$ and π; this function is `signum(x)` in Maple).

(b) Triangular wave: $f(x) = |x| = \begin{cases} -x & -\pi \leq x < 0 \\ x & 0 \leq x < \pi \end{cases}$.

For both (a) and (b), also (optional, not for this exam) plot the *periodic extension* of f, that is, the function repeated 2π-periodically, on $[-3\pi, 3\pi]$, and use Maple to plot the Fourier polynomials to the Nth harmonic on $[-3\pi, 3\pi]$ with $N = 1$, 3, 9 and 21, to see the convergence when one uses more terms. Note that in (a) we see an "overshoot" and oscillations near the discontinuities in (the periodic extension of) f (Gibbs phenomenon), while in (b), we do not see such oscillations, since (the periodic extension of) f is continuous.