MATH 252-3 **Spring 2005**

Vector Calculus

Homework Set 6 Due Wednesday, 2 March 2005

Course Web Site: `http://www.math.sfu.ca/~ralfw/math252/`

Textbook: **Davis and Snider** "*Introduction to Vector Analysis*"

Midterm Exam Friday 25 February
Due to the midterm exam, there will be no homework set due on Wednesday 23 February; the problems listed below cover two weeks' worth of material, and will be due on Wednesday 2 March. However, for midterm exam study purposes, you should aim to work on most of these problems before the exam (however, only parts (a) and (b) of the extra problem may be relevant to the exam). I will let you know on Monday 21 February which sections will be covered on the midterm.

Reading: Sections 3.5–3.11
Please read about some physical applications of the Laplacian operator (see Section 3.6 and lecture notes pp.77–79).

Problems to study (for practice; you do not need to hand these in):

- Section 3.5 (pp.135–136): 4, 8
- Section 3.6 (p.140): 8
- Section 3.8 (pp.150–151): 5, 7, 8, 9, 10(a)–(e)
- Section 3.9 (p.153): 4
- Section 3.10 (pp.169–170): 3, 6, 7, 11, 13, 14, 15
- Section 3.11 (pp.180–182): 4, 6, 8, 10

Problems to hand in:

- *Section 3.6 (p.140): 3, 4, 5*
- *Section 3.8 (pp.150–151): 2 (use tensor notation except for (3.33)), 10(f)–(h), 12, 13, 14*
- *Section 3.9 (p.153): 1, 5, 7*
- *Section 3.10 (pp.169–170): 2, 8, 9, 10, 12, 16*
- *Section 3.11 (pp.180–182): 3, 7, 9*

Extra problem (to hand in)

1. *Maxwell's Equations and Electromagnetic Waves*
   In this problem, we will demonstrate the usefulness of vector identities and some additional physical applications of the div, curl and Laplacian operators by investigating the differential form of *Maxwell's Equations*, which form the basis of the theory of electromagnetism: their solutions describe the phenomenology of electrostatics, magnetostatics, and dynamic phenomena such as electromagnetic waves.

   Maxwell's equations are a system of linear partial differential equations for the electric field **E** and magnetic field **B**. In SI units (MKS units: that is, using the metre, kg, second, and

Ampere as basic units), in the absence of magnetic or polarizable media

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

Here ρ is the charge density (charge per unit volume: units $\mathrm{C/m^3}$) and $\mathbf{J}$ is the current density (current per unit area: units $\mathrm{A/m^2}$, where C = Coulomb, A = Ampere = Coulomb/second). Also, ϵ_0 = permittivity of free space $= 8.8542 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$, (N = Newtons = $\mathrm{kg\,m\,s^{-2}}$) and μ_0 = permeability of free space $= 4\pi \times 10^{-7} \mathrm{NA}^{-2}$.

The first equation (1) above is the differential form of Gauss' Law for electricity; (2) is Gauss' Law for magnetism (no magnetic monopoles); (3) represents Faraday's Law of induction, while equation (4) is Ampère's Law. [In dielectric, magnetic or polarizable materials, Maxwell's equations are modified to take into account the electric permittivity, magnetic permeability and polarization of the medium.]

The mathematical and physical consequences of Maxwell's equations are profound, and represent a major scientific achievement of the 19th century. For now, we will explore how vector identities can be used to deduce the conservation of charge and the existence of electromagnetic waves, and study some basic properties of these waves.

(a) From Maxwell's equations (1) and (4), derive the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

which represents the conservation of charge (where ρ is charge density, and $\mathbf{J}$ is the current density, or charge flux density).
[Hint: Take $\partial/\partial t$ of (1); use the fact that partial derivatives with respect to space and time variables commute (equality of mixed partial derivatives), so that $\partial/\partial t(\nabla \cdot \mathbf{E}) = \nabla \cdot (\partial \mathbf{E}/\partial t)$; substitute for $\partial \mathbf{E}/\partial t$ from (4) and use a vector identity.]

(b) Maxwell's equations in a vacuum, in the absence of any charges or currents ($\rho = 0$, $\mathbf{J} = \mathbf{0}$), are given by:

$$\nabla \cdot \mathbf{E} = 0 \tag{5}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

Show that the electric field $\mathbf{E}$ satisfies the *wave equation*

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}\ , \tag{9}$$

where $c^2 = 1/\mu_0 \epsilon_0$. Similarly, verify that the magnetic field $\mathbf{B}$ also satisfies (9).
(This vector equation means that each *component* E_i, B_i of $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation $\partial^2 u/\partial t^2 = c^2 \nabla^2 u$.)

[Hint: Take $\partial/\partial t$ of (8), commute the spatial and temporal derivatives, and substitute from (7); use a suitable vector identity.]

The constant $c = 1/\sqrt{\mu_0\epsilon_0} \approx 2.99 \times 10^8 \mathrm{m\,s^{-2}}$ is the speed of light! By this calculation, Maxwell realized that *light is an electromagnetic wave, propagating at speed c.*

We will explore some of the details of Maxwell's calculation. To see why (9) is a wave equation, assume single-frequency solutions to (5)–(8) for the electric and magnetic field of the form

$$\mathbf{E} = [A\cos(\omega t - \mathbf{k}\cdot\mathbf{R}) + B\sin(\omega t - \mathbf{k}\cdot\mathbf{R})]\mathbf{u}_E\ , \tag{10}$$

$$\mathbf{B} = [C\cos(\omega t - \mathbf{k}\cdot\mathbf{R}) + D\sin(\omega t - \mathbf{k}\cdot\mathbf{R})]\mathbf{u}_B\ . \tag{11}$$

Here: $\mathbf{u}_E$ and $\mathbf{u}_B$ are suitable constant unit vectors indicating the *direction* of the fields $\mathbf{E}$ and $\mathbf{B}$, respectively;
$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the position vector;
ω is a constant, the angular frequency of the wave ($\omega = 2\pi\nu$, where ν is the *frequency*);
$\mathbf{k} = k_1\hat{\mathbf{i}} + k_2\hat{\mathbf{j}} + k_3\hat{\mathbf{k}}$ is a constant vector, whose direction indicates the direction of propagation of the wave, and whose magnitude is $|\mathbf{k}| = 2\pi/\lambda$, where λ is the *wavelength*;
and A, B, C and D are constants.
[Note: you should be careful to distinguish between $\mathbf{k}$, the wave propagation vector with magnitude $2\pi/\lambda$, and $\hat{\mathbf{k}}$, the unit vector parallel to the z-axis.]

(c) Show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equation (9) provided $c = \omega/|\mathbf{k}|$.

[You need consider only $\mathbf{E}$. Compute $\partial^2\mathbf{E}/\partial x^2$ to show that $\partial^2\mathbf{E}/\partial x^2 = -k_1^2\mathbf{E}$; by a similar calculation for the y and z derivatives, show that $\nabla^2\mathbf{E} = -|\mathbf{k}|^2\mathbf{E}$. Similarly, compute $\partial^2\mathbf{E}/\partial t^2$, and show $\partial^2\mathbf{E}/\partial t^2 = -\omega^2\mathbf{E}$. Thus show that the expression (10) for $\mathbf{E}$ satisfies the wave equation (9) provided ω and $|\mathbf{k}|$ are related by $\omega = c|\mathbf{k}|$. Some notes regarding the differentiation of $\mathbf{E}$ and $\mathbf{B}$ are on the course web page.]

(d) Explain why the fields $\mathbf{E}$ and $\mathbf{B}$ defined in (10)–(11) represent *waves* propagating in the direction of $\mathbf{k}$, with *wave speed* $c = \nu\lambda = \omega/|\mathbf{k}|$.

[You need consider only $\mathbf{E}$. Show that $\mathbf{E}$ takes the same value at time $t = 0$, position vector $\mathbf{R}_0$, and at time t, position vector $\mathbf{R}$, provided that $\omega t - \mathbf{k}\cdot(\mathbf{R} - \mathbf{R}_0) = 0$. Thus show that if $\mathbf{R} - \mathbf{R}_0$ is parallel to $\mathbf{k}$, then $\mathbf{E}$ is constant along points with position vectors $\mathbf{R} = \mathbf{R}(t)$ satisfying $|\mathbf{R} - \mathbf{R}_0| = \omega t/|\mathbf{k}| = ct$; and hence interpret c as the wave speed.]

(e) Show that $\nabla\cdot\mathbf{E} = 0$ implies $\mathbf{k}\cdot\mathbf{u}_E = 0$. Thus show that (5) and (10) imply that that the electric field vector $\mathbf{E}$ is perpendicular to the direction of propagation, $\mathbf{k}\cdot\mathbf{E} = 0$. Similarly, show that $\mathbf{k}$ is perpendicular to $\mathbf{u}_B$ and thus to $\mathbf{B}$.
[See the notes on the course web page.]

(f) Substitute $\mathbf{E}$ and $\mathbf{B}$ into (7) to show that $\mathbf{k}\times\mathbf{u}_E$ is parallel to $\mathbf{u}_B$, and thus that $\mathbf{u}_E\cdot\mathbf{u}_B = 0$; thus the electric and magnetic fields are mutually perpendicular.
Also use (7) to find a relation between A and C, and a relation between B and D. (You would obtain the same result using (8) instead of (7)).

In summary, for an electromagnetic wave, such as light (or radio waves, infrared or ultraviolet radiation, X-rays or gamma rays) $\mathbf{k}$, $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular vectors; the changing electric field induces a changing magnetic field (according to (8)) which in turn induces an electric field (by (7)), and together these fields propagate at speed c in the direction of $\mathbf{k}$ while remaining mutually perpendicular to each other and to $\mathbf{k}$.

For more information on the theory of electromagnetism, see Appendix D of the textbook by Davis and Snider, the book by Shey in the library reserves, or any one of many suitable physics references.