Orthogonality

The idea of orthogonality plays a central role in many branches of mathematics and its applications.

Vector Space

A vector space V (over the reals) consists of elements ("vectors") eg $\vec{u}, \vec{v} \in V$

together with a chosen field of scalars $s, t \in \mathbb{R}$

(we consider only real-valued vectors, functions etc ; in general could have eg complex numbers, but our scalars will be real numbers)

such that V is closed under addition and scalar multiplication

ie $\vec{u} + \vec{v} \in V$, $s\vec{u} \in V$ for all $\vec{u}, \vec{v} \in V$, $s, t \in \mathbb{R}$

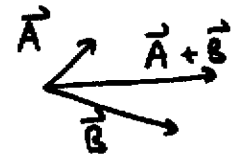
$(\Rightarrow s\vec{u} + t\vec{v} \in V)$

with the properties:

- Associativity: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- Commutativity: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- Zero vector $\vec{0}$: $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
- Inverse vector: $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$
- Distributivity: $s(\vec{u} + \vec{v}) = s\vec{u} + s\vec{v}$
  $(s+t)\vec{u} = s\vec{u} + t\vec{u}$
- Zero scalar 0: $0\,\vec{u} = \vec{0}$
  Unit scalar 1: $1\,\vec{u} = \vec{u}$

Examples:

- Vectors in $\mathbb{R}^2$:



  Vectors in $\mathbb{R}^3$

  Vectors in $\mathbb{R}^n$ (n-tuples), $\vec{A}, \vec{B}, \ldots \in \mathbb{R}^n$

- Set of quadratic polynomials (degree ≤ 2)

  $$a_0 + a_1 x + a_2 x^2 \quad , \quad a_0, a_1, a_2 \in \mathbb{R}$$

  (sum of two quadratic polynomials is a quadratic polynomial; so is any multiple of a quadratic polynomial by a real number)

  More generally: set P_n of polynomials of degree $\leq n$:

  $$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \quad , \quad a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$$

- Set of continuous functions on $[-1,1]$ ← more generally, $[a,b]$

  $$f, g \in C[-1,1] \quad , \quad s, t \in \mathbb{R}$$

  $$\Rightarrow sf + tg \in C[-1,1]$$

- Set of functions on $[a,b]$ vanishing at the endpoints

  $$f(a) = f(b) = 0$$

Basis

A set of linearly independent elements of V so that each element ("vector") of V can be written as a linear combination of basis elements

Linear combination:

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. $\vec{u}$ is a linear combination of the $\{\vec{v}_i\}$ if there are scalars $t_1, t_2, \ldots, t_n \in \mathbb{R}$ so that $\vec{u} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_n\vec{v}_n = \sum_{i=1}^{n} t_i \vec{v}_i$

Linear independence:

The vectors $\{\vec{v}_i\}$ are linearly independent if the only solution of $t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_n\vec{v}_n = \sum_{i=1}^{n} t_i\vec{v}_i = \vec{0}$

is $t_1 = t_2 = \cdots = t_n = 0$

(no nontrivial linear combination adds to $\vec{0}$; equivalently, none of the $\vec{v}_i$ is a linear combination of the others)

– else they are linearly dependent.

Fact: If one basis of the vector space V has a finite number N of elements, then every basis has N elements; then V is an N-dimensional vector space.

eg in $\mathbb{R}^2$ (a two-dimensional space: every basis has 2 elements)

$\hat{\imath}, \hat{\jmath}$ forms a basis

Another basis: $\vec{a}_1 = \hat{\imath} + 2\hat{\jmath}$, $\vec{a}_2 = 3\hat{\imath} - \hat{\jmath}$

eg in the set of quadratic polynomials P_2:

$\{1, x, x^2\}$ form a basis

(since every quadratic polynomial is $a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$)

$\Rightarrow$ P_2 is a 3-dimensional vector space.

Similarly P_n, the space of n^{th} degree polynomials, is an $(n+1)$-dimensional vector space, with basis $\{1, x, \ldots, x^n\}$

Inner Product (Scalar product, "Dot Product")

of two elements ("vectors") of V
is a scalar (real), denoted $(\vec{u}, \vec{v})$ or $\vec{u} \cdot \vec{v}$

conventional notation for functions

convention for vectors $\in \mathbb{R}^n$

satisfying (for real vector space)

- $(\vec{u}, \vec{v}) \in \mathbb{R}$
- $(\vec{u}, \vec{v}) = (\vec{v}, \vec{u})$
- Linearity $(s\vec{u}_1 + t\vec{u}_2, \vec{v}) = s(\vec{u}_1, \vec{v}) + t(\vec{u}_2, \vec{v})$
- $(\vec{u}, \vec{u}) \geq 0$
  $(\vec{u}, \vec{u}) = 0 \iff \vec{u} = \vec{0}$ in V

ie $(\vec{u}, \vec{u}) > 0$ for all nonzero vectors $\vec{u} \in V$

eg for vectors in $\mathbb{R}^2$: $\vec{v}_1 = a_1 \underset{\hat{e}_1}{\hat{\imath}} + a_2 \underset{\hat{e}_2}{\hat{\jmath}}$, $\vec{v}_2 = b_1 \hat{\imath} + b_2 \hat{\jmath}$

$$\Rightarrow (\vec{v}_1, \vec{v}_2) = \vec{v}_1 \cdot \vec{v}_2 = a_1 b_1 + a_2 b_2$$

vectors in $\mathbb{R}^n$: $\vec{v} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + \dots + a_n \hat{e}_n = \sum_{i=1}^{n} a_i \hat{e}_i$

$\vec{w} = b_1 \hat{e}_1 + b_2 \hat{e}_2 + \dots + b_n \hat{e}_n$

$\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$: orthonormal basis

eg Cartesian basis

$$\Rightarrow \vec{v} \cdot \vec{w} = a_1 b_1 + \dots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Norm

Given an inner product, we can define the "length"/ "magnitude" of a vector – the norm:

$$\|\vec{v}\| = [\vec{v} \cdot \vec{v}]^{1/2} = [(\vec{v}, \vec{v})]^{1/2}$$

Note: $\|\vec{v}\| \in \mathbb{R}$

$\|\vec{v}\| > 0$ if $\vec{v} \neq \vec{0}$, $\|\vec{0}\| = 0$

eg for vectors in $\mathbb{R}^n$, $\|\vec{v}\| = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$ (Pythagoras' Theorem)

$|\vec{v}|$

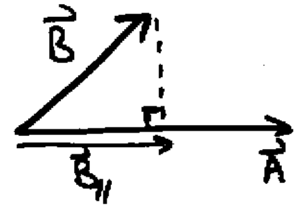
Euclidean norm

Orthogonality

Two nonzero vectors $\vec{u}, \vec{v} \in V$ are orthogonal if $(\vec{u}, \vec{v}) = 0$.

Consequence of Orthogonality:

Recall: projection



$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

$$\vec{B}_{//} = \frac{\vec{B}\cdot\vec{A}}{|\vec{A}|^2}\,\vec{A} = \underbrace{(\vec{B}\cdot\hat{A})}_{\text{comp}_{\vec{A}}\vec{B}}\,\hat{A} = \text{proj}_{\vec{A}}\vec{B}$$

Suppose an element of V is expanded in terms of orthogonal basis elements; then we can easily find the expansion coefficients by projection
(ie taking inner|dot products)

eg for vectors in $\mathbb{R}^3$:
Let $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be any set of mutually orthogonal vectors $(\vec{b}_i\cdot\vec{b}_j = 0 \text{ for } i \neq j)$

$\Rightarrow$ they form a basis of $V = \mathbb{R}^3$ (independent).

For any vector $\vec{u}$, we can write

$$\vec{u} = t_1\vec{b}_1 + t_2\vec{b}_2 + t_3\vec{b}_3$$

Since the $\{\vec{b}_i\}$ are orthogonal, find the coefficients t_i by projection:

$$\vec{u}\cdot\vec{b}_1 = t_1\,\vec{b}_1\cdot\vec{b}_1 + t_2\underbrace{\vec{b}_2\cdot\vec{b}_1}_{=0} + t_3\underbrace{\vec{b}_3\cdot\vec{b}_1}_{=0}$$
$$= t_1\,\vec{b}_1\cdot\vec{b}_1 + 0 + 0$$

$$\Rightarrow t_1 = \frac{\vec{u}\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1} \qquad \text{Similarly } t_2 = \frac{\vec{u}\cdot\vec{b}_2}{\vec{b}_2\cdot\vec{b}_2}, \quad t_3 = \frac{\vec{u}\cdot\vec{b}_3}{\vec{b}_3\cdot\vec{b}_3}$$

and $$\vec{u} = \sum_{j=1}^{3} \underbrace{\left(\frac{\vec{u}\cdot\vec{b}_j}{\vec{b}_j\cdot\vec{b}_j}\right)}_{t_j} \vec{b}_j = \text{proj}_{\vec{b}_1}\vec{u} + \text{proj}_{\vec{b}_2}\vec{u} + \text{proj}_{\vec{b}_3}\vec{u}$$

Note: from an orthogonal basis, we can construct an orthonormal basis (of mutually orthogonal unit vectors)

by $\hat{e}_i = \dfrac{\vec{b}_i}{|\vec{b}_i|}$, then $\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = \delta_{ij}$

In this case, the expansion coefficients are:

$$\vec{u} = s_1\hat{e}_1 + s_2\hat{e}_2 + s_3\hat{e}_3$$

$$\Rightarrow \vec{u}\cdot\hat{e}_1 = s_1 \underbrace{\hat{e}_1\cdot\hat{e}_1}_{=1} + 0 + 0 = s_1, \quad s_2 = \vec{u}\cdot\hat{e}_2$$
$$s_3 = \vec{u}\cdot\hat{e}_3$$

In general, if $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_N\}$ are N mutually orthogonal vectors:

$$\vec{u} = t_1\vec{b}_1 + \ldots + t_N\vec{b}_N = \sum_{j=1}^{N} t_j\vec{b}_j$$

$$\rightarrow \vec{u}\cdot\vec{b}_i = \sum_{j=1}^{N} t_j \underbrace{\vec{b}_j\cdot\vec{b}_i}_{=0 \text{ if } i\neq j} = t_i\, \vec{b}_i\cdot\vec{b}_i$$

← projection of $\vec{u}$ onto i^{th} basis element, using orthogonality

$$\Rightarrow t_i = \frac{\vec{u}\cdot\vec{b}_i}{\vec{b}_i\cdot\vec{b}_i}, \quad i = 1, \ldots, N$$

Gram-Schmidt Orthogonalization

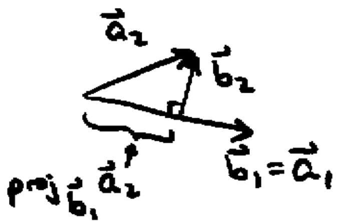
Given a set of <u>independent</u> elements of V, we can construct an <u>orthogonal</u> set by the Gram-Schmidt process

eg for vectors in R^N

- Assume $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\}$ are independent.

Start with $\vec{b}_1 = \vec{a}_1$

Now find $\vec{b}_2 \in \text{span}\{\vec{a}_1, \vec{a}_2\}$, orthogonal to $\vec{b}_1$, by starting with $\vec{a}_2$ and subtracting the component in the direction of $\vec{b}_1$



$$\vec{b}_2 = \vec{a}_2 - \text{proj}_{\vec{b}_1}\vec{a}_2 = \vec{a}_2 - \frac{\vec{a}_2\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1}\vec{b}_1$$

(Check: $\vec{b}_2\cdot\vec{b}_1 = \vec{a}_2\cdot\vec{b}_1 - \left(\frac{\vec{a}_2\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1}\right)\vec{b}_1\cdot\vec{b}_1 = 0$)

Continue similarly:

$$\vec{b}_3 = \vec{a}_3 - \text{proj}_{\vec{b}_1}\vec{a}_3 - \text{proj}_{\vec{b}_2}\vec{a}_3 = \vec{a}_3 - \sum_{j=1}^{2}\frac{\vec{a}_3\cdot\vec{b}_j}{\vec{b}_j\cdot\vec{b}_j}\vec{b}_j$$

$$\vdots$$

$$\vec{b}_k = \vec{a}_k - \underbrace{\sum_{j=1}^{k-1}\underbrace{\left(\frac{\vec{a}_k\cdot\vec{b}_j}{\vec{b}_j\cdot\vec{b}_j}\right)\vec{b}_j}_{\text{proj}_{\vec{b}_j}\vec{a}_k}}_{\text{projection of } \vec{a}_k \text{ onto subspace spanned by } \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_{k-1}\}}$$

[Check that $\vec{b}_k$ is orthogonal to $\vec{b}_i$ $\quad 1 \le i < k$: (assume $\{\vec{b}_1, \dots \vec{b}_{k-1}\}$ orthogonal)

$$\vec{b}_k\cdot\vec{b}_i = \vec{a}_k\cdot\vec{b}_i - \sum_{j=1}^{k-1}\left(\frac{\vec{a}_k\cdot\vec{b}_j}{\vec{b}_j\cdot\vec{b}_j}\right)\underbrace{\vec{b}_j\cdot\vec{b}_i}_{=0,\ j\neq i}$$

$$= \vec{a}_k\cdot\vec{b}_i - \frac{\vec{a}_k\cdot\vec{b}_i}{\vec{b}_i\cdot\vec{b}_i}\vec{b}_i\cdot\vec{b}_i = 0$$]

eg given $\vec{a}_1 = \hat{i} + 2\hat{j}$, $\vec{a}_2 = 3\hat{i} - \hat{j}$:

Gram-Schmidt: $\vec{b}_1 = \vec{a}_1 = \hat{i} + 2\hat{j}$

$$\vec{b}_2 = \vec{a}_2 - \text{proj}_{\vec{b}_1}\vec{a}_2 = (3\hat{i}-\hat{j}) - \frac{\vec{a}_2\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1}(\hat{i}+2\hat{j})$$

$$= (3\hat{i}-\hat{j}) - \frac{1}{5}(\hat{i}+2\hat{j}) = \frac{14}{5}\hat{i} - \frac{7}{5}\hat{j}$$

$$= \frac{7}{5}(2\hat{i}-\hat{j})$$

- clearly $\vec{b}_1 \cdot \vec{b}_2 = 0$.

Orthonormal set: $\hat{e}_1 = \frac{\vec{b}_1}{|\vec{b}_1|} = \frac{1}{\sqrt{5}}(\hat{i}+2\hat{j})$, $\hat{e}_2 = \frac{\vec{b}_2}{|\vec{b}_2|} = \frac{1}{\sqrt{5}}(2\hat{i}-\hat{j})$

This general approach is appropriate to any vector space V with an inner product $(\cdot,\cdot)$:

- construct orthogonal basis by Gram-Schmidt process
- expand in orthogonal basis by projection (taking inner products).

Inner Products for Functions :

A vector $\vec{A}$ in $\mathbb{R}^N$ (N-dimensional space) is completely described by N values

A_i , $i = 1, .., N$ (the components of $\vec{A}$ wrt. an orthonormal (Cartesian) basis).

Suppose the number of dimensions $\to$ infinity

vectors $\to$ (continuous) functions (assume)

- a function f is described by its values $f(x)$ at infinitely many points x.

Inner product of vectors $\vec{A}$, $\vec{B}$ is $\vec{A}\cdot\vec{B} = \sum_{i=1}^{N} A_i B_i$

As $N\to\infty$, the sum in the inner product $\longrightarrow$ integral.

This motivates:

Consider functions on an interval $[a,b]$. (real-valued)

Define an <u>inner product</u> for functions:

$$\boxed{(f,g) = \int_a^b f(x)\,g(x)\,dx}$$

(Check that this satisfies the requirements for inner products: linearity; $(f,f)\geq 0$, $(f,g)=(g,f)$
$(f,f)=\int_a^b f^2 dx = 0 \iff f(x)\equiv 0$.)

Functions f, g are <u>orthogonal</u> on $[a,b]$

if $(f,g) = \int_a^b f(x)\,g(x)\,dx = 0$.

Sometimes it is convenient to introduce a <u>weight function</u>

$w(x) > 0$ (sometimes one can allow $w(x)\geq 0$,
or $\rho(x)$* with $w=0$ possibly at isolated points)

– then define a weighted inner product

$$(f,g)_w = \int_a^b f(x)\,g(x)\,w(x)\,dx$$

Legendre Polynomials

Consider polynomials on $[-1,1]$ (ie $a=-1$, $b=1$)
with inner product $(f,g) = \int_{-1}^{1} f(x)\, g(x)\, dx$.

Basis of polynomials : $1, x, x^2, x^3, \ldots$

(eg the polynomials $1, x, x^2, x^3$ are linearly independent, form a basis for P_3, the 4-dimensional space of cubic polynomials, polynomials of degree ≤ 3)

Note: the polynomials $1, x$ are orthogonal :

$$(1,x) = \int_{-1}^{1} 1 \cdot x \, dx = \tfrac{1}{2} x^2 \Big|_{-1}^{1} = 0$$

but in general, this basis is not orthogonal w.r.t. this inner product eg $1, x^2$ are not orthogonal :

$$(1,x^2) = \int_{-1}^{1} 1 \cdot x^2 \, dx = \tfrac{1}{3} x^3 \Big|_{-1}^{1} = 2/3 \neq 0.$$

Apply Gram-Schmidt process to

$f_0 = 1,\ f_1 = x,\ f_2 = x^2,\ f_3 = x^3, \ldots$

to obtain an orthogonal set (w.r.t the given inner product):

- $p_0(x) = f_0(x) = 1$

- $$p_1(x) = f_1(x) - \frac{(f_1, p_0)}{(p_0, p_0)} p_0(x)$$
  $$= x - \frac{(x,1)}{(1,1)} 1$$
  $$= x$$

$(1,1) = \int_{-1}^{1} 1^2 dx = 2$

$(x,1) = \int_{-1}^{1} x \cdot 1 dx = 0$

- $$p_2(x) = f_2(x) - \sum_{j=0}^{1} \frac{(f_2, p_j)}{(p_j, p_j)} p_j(x)$$

$$= x^2 - \frac{(x^2,1)}{(1,1)} 1 - \frac{(x^2,x)}{(x,x)} x$$

$$= x^2 - \frac{2/3}{2} \cdot 1 - \frac{0}{2/3} x$$

$$= x^2 - 1/3$$

$(x^2, 1) = \int_{-1}^{1} x^2 \cdot 1 \, dx = \frac{2}{3}$

$(x^2, x) = \int_{-1}^{1} x^2 \cdot x \, dx = 0$

$(x, x) = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$(x^3, 1) = \int_{-1}^{1} x^3 \cdot 1 \, dx = 0$

$(x^3, x) = \int_{-1}^{1} x^3 \cdot x \, dx = \frac{2}{5}$

$(x^3, p_2) = \int_{-1}^{1} x^3(x^2 - 1/3) dx = 0$

- $$p_3(x) = f_3(x) - \sum_{j=0}^{2} \frac{(f_3, p_j)}{(p_j, p_j)} p_j(x)$$

$$= x^3 - \frac{(x^3,1)}{(1,1)} 1 - \frac{(x^3,x)}{(x,x)} x - \frac{(x^3, p_2)}{(p_2,p_2)} p_2(x)$$

$$= x^3 - 0 - \frac{2/5}{2/3} x - 0$$

$$= x^3 - \frac{3}{5} x$$

⋮

- can continue this process to obtain higher degree polynomials.

<u>Note</u>:
- $p_n(x)$ is a polynomial of degree n
- The polynomials $\{p_n(x)\}$ are orthogonal w.r.t. the given inner product:

$$(p_m, p_n) = \int_{-1}^{1} p_m(x) p_n(x) dx = 0 \quad \text{if } m \neq n$$

- $p_n(x)$ is even if n is even, odd if n odd

ie $p_n(-x) = (-1)^n p_n(x)$

Usually normalize the polynomials to obtain $P_n(x) = a_n p_n(x)$ so that $P_n(1) = 1$

- gives the Legendre polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$
$$\vdots$$

· Each $P_k(x)$ is a linear combination of x^j, $j \le k$, $P_k(x)$ independent

$\Leftrightarrow$ x^k is a linear combination of $P_j(x)$, $j \le k$

$\{P_0(x), P_1(x), \ldots, P_n(x)\}$ forms a basis (orthogonal!) for $P_n[-1,1]$, the vector space of polynomials of degree $\le n$ on $[-1,1]$

(n+1)-dimensional
— any (n+1) independent elements form a basis

· If $q(x)$ is a polynomial of degree n, then

expansion in orthogonal basis $\rightarrow$
$$q(x) = \sum_{j=0}^{n} c_j P_j(x) = c_0 P_0(x) + \ldots + c_n P_n(x)$$

and the coefficients c_j can be found by projection:

$$(q, P_i) = \sum_{j=0}^{n} c_j \underbrace{(P_j, P_i)}_{=0 \text{ if } i \ne j} = c_i (P_i, P_i)$$

$$\Rightarrow c_i = \frac{(q, P_i)}{(P_i, P_i)} = \frac{\int_{-1}^{1} q(x) P_i(x)\,dx}{\int_{-1}^{1} P_i^2(x)\,dx}$$

Approximation by Legendre Polynomials

If $f(x)$ is a function on $[-1,1]$, then

$$F_n(x) = \sum_{j=0}^{n} \frac{(f, P_j)}{(P_j, P_j)} P_j(x)$$

is an approximation of f by polynomials of degree $\leq n$.

eg $n=0$:

$$F_0(x) = \frac{(f, P_0)}{(P_0, P_0)} P_0(x) = \frac{1}{2}\left[\int_{-1}^{1} 1 \cdot f(x)\,dx\right] \cdot 1$$

$P_0 = 1$
$(P_0, P_0) = 2$

$$= \underbrace{\frac{1}{2}\int_{-1}^{1} f(x)\,dx}_{c_0}$$

: average of f on $[-1,1]$

$n=1$:

$$F_1(x) = \frac{(f, P_0)}{(P_0, P_0)} P_0(x) + \frac{(f, P_1)}{(P_1, P_1)} P_1(x)$$

$P_1 = x$
$(P_1, P_1) = \int_{-1}^{1} x^2 dx = 2/3$

$$= F_0(x) + \underbrace{\frac{3}{2}\left[\int_{-1}^{1} x f(x)\,dx\right]}_{c_1} x$$

$$= c_0 + c_1 x$$

- a linear approximation of f over the entire interval $[-1,1]$.

[Note: one can show

$$(P_n, P_n) = \int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \ldots$$

normalization

$$\Rightarrow (P_m, P_n) = \frac{2}{2n+1} \delta_{mn}$$
]

Comparison with Taylor Polynomials

Recall: Linear approximation of f near $x = x_0$:

Approximate $f(x)$ by $T_1(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}$

tangent line to graph of f at $x = x_0$

- best linear approximation to $f(x)$ near x_0

Compare: the approximation by Legendre polynomials is an approximation over the entire interval $[-1,1]$

Similarly: n^{th} degree Taylor polynomial

$$T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!}$$

- a good local approximation to f, near x_0

Note: Taylor polynomial uses information about f and its derivatives at a single point x_0

- good locally, not necessarily globally.

Legendre polynomial approximation

$$F_n(x) = \sum_{j=0}^{n} \frac{\int_{-1}^{1} f(x) P_j(x)\,dx}{\int_{-1}^{1} P_j(x)^2\,dx} P_j(x)$$

uses information about $f(x)$ on the entire interval $[-1,1]$ (from integrals of f): a global approximation

Legendre polynomials have many other properties and applications, such as:

Facts
- Rodrigues' Formula $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \quad n=0,1,2,\dots$$

- Recurrence relations – a Legendre polynomial at one point x can be expressed by neighbouring Legendre polynomials at x

  eg $$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n=1,2,3,\dots$$
  (allows one to compute all $P_n(x)$ from $P_0=1$, $P_1=x$)
  – and several other similar relations

- Differential equation $P_n(x)$ solves Legendre's differential equation
  $$(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n = 0$$

– numerous applications, such as in numerical analysis (integration formulas, approximation) and differential equations, especially to the solution of problems (eg Laplace's equation) in spherical coordinates (with $x = \cos\phi$)

Given a different interval $[a,b]$ and/or a different weight function $p(x)$ (inner product $(f,g)_p = \int_a^b p(x)f(x)g(x)\,dx$) we get a different set of orthogonal polynomials, with different properties and applications

eg on $[-1,1]$, $p(x) = \frac{1}{\sqrt{1-x^2}}$: Chebyshev polynomials

$(-\infty,\infty)$, $p(x) = e^{-x^2}$: Hermite polynomials

$[0,\infty)$, $p(x) = e^{-x}$: Laguerre polynomials

Trigonometric Approximation and Fourier Series

Many processes in science and engineering are periodic, or have oscillatory properties (eg heartbeat, waves, signals)

– would like to approximate them by periodic functions

Basic building blocks: Sinusoids (sine, cosine) with different frequencies form the basic oscillatory functions (simple harmonic oscillations):

the functions

$$1, \quad \cos x, \sin x, \quad \cos 2x, \sin 2x, \ldots, \quad \cos kx, \sin kx, \ldots$$

cos 0x

are all 2π - periodic.

We wish to approximate a function $f(x)$ with a superposition of trigonometric functions

on the interval $[-\pi, \pi]$ of length 2π

$$f(x) \approx F_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$

$$\Rightarrow F_n(x) = a_0 + \sum_{k=1}^{n} \left[a_k \cos kx + b_k \sin kx \right]$$

$F_n(x)$: trigonometric polynomial of degree n (Fourier polynomial)

a_k, b_k: Fourier coefficients

How do we find the coefficients a_k, b_k?

Fundamental property:

The functions

$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos kx, \sin kx, \ldots$

are orthogonal on $[-\pi, \pi]$, with respect to the inner product $\quad (f,g) = \int_{-\pi}^{\pi} f(x)\,g(x)\,dx$.

Orthogonality conditions:

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 2\pi & m = n = 0 \\ \pi & m = n \neq 0 \\ 0 & m \neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases} = \pi \delta_{mn}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \qquad \begin{aligned} m &= 0, 1, 2, 3, \ldots \\ n &= 1, 2, 3, \ldots \end{aligned}$$

Check using trigonometric identities

eg $\quad \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$

$$= \begin{cases} (\text{for } m \neq n,\ m,n = 1,2,3,\ldots) \\ \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right]\Big|_{-\pi}^{\pi} = 0 \\ (\text{for } m = n = 1,2,3,\ldots) \\ \frac{1}{2}\left[x - \frac{1}{m+n}\sin(m+n)x\right]\Big|_{-\pi}^{\pi} = \pi \end{cases}$$

Due to the orthogonality, the expansion

$$f(x) \approx F_n(x) = a_0 + \sum_{k=1}^{n} [a_k \cos kx + b_k \sin kx]$$

is an orthogonal expansion, and we can find the Fourier coefficients by projection (taking inner products):

inner product with $1 = \cos 0x$

$$(f,1) = a_0(1,1) + \sum_{k=1}^{n} [a_k \underbrace{(\cos kx, 1)}_{=0} + b_k \underbrace{(\sin kx, 1)}_{=0}]$$

$$= a_0(1,1)$$

$$\Rightarrow a_0 = \frac{(f,1)}{(1,1)} = \frac{\int_{-\pi}^{\pi} 1 \cdot f(x)\,dx}{\int_{-\pi}^{\pi} 1^2\,dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$1 \le m \le n$

$$(f, \cos mx) = a_0 \underbrace{(1, \cos mx)}_{=0} + \sum_{k=1}^{n} [a_k \underbrace{(\cos kx, \cos mx)}_{=0 \text{ for } k \neq m} + b_k \underbrace{(\sin kx, \cos mx)}_{=0}]$$

$$= a_m (\cos mx, \cos mx)$$

$$\Rightarrow a_m = \frac{(f, \cos mx)}{(\cos mx, \cos mx)} = \frac{\int_{-\pi}^{\pi} f(x) \cos mx\,dx}{\int_{-\pi}^{\pi} \cos^2 mx\,dx \leftarrow \pi}$$

$1 \le m \le n$

$$(f, \sin mx) = a_0 \underbrace{(1, \sin mx)}_{=0} + \sum_{k=1}^{n} [a_k \underbrace{(\cos kx, \sin mx)}_{=0} + b_k \underbrace{(\sin kx, \sin mx)}_{=0 \text{ for } m \neq k}]$$

$$= b_m (\sin mx, \sin mx)$$

$$\Rightarrow b_m = \frac{(f, \sin mx)}{(\sin mx, \sin mx)} = \frac{\int_{-\pi}^{\pi} f(x) \sin mx\,dx}{\int_{-\pi}^{\pi} \sin^2 mx\,dx \leftarrow \pi}$$

Thus the coefficients in the trigonometric (Fourier) expansion

$$f(x) \approx F_n(x) = a_0 + \sum_{k=1}^{n} \left[a_k \cos kx + b_k \sin kx \right]$$

are given by:

<u>Fourier coefficients</u>

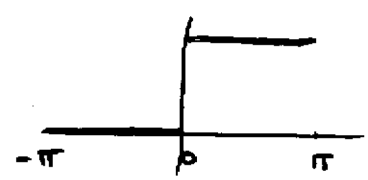
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \leftarrow \text{average of } f \text{ on } [-\pi, \pi]$$

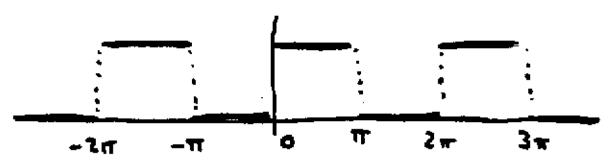
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

eg square wave (period 2π)

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$



(2π-periodic extension 

)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{0}^{\pi} \cos kx\,dx = \frac{1}{\pi k} \sin kx \Big|_0^{\pi} = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{0}^{\pi} \sin kx\,dx = -\frac{1}{\pi k} \cos kx \Big|_0^{\pi}$$

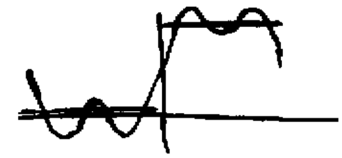
$$= \frac{1}{\pi k}\left(1 - \cos k\pi\right) = \frac{1}{\pi k}\left(1 - (-1)^k\right) = \begin{cases} 0 & k \text{ even} \\ \frac{2}{\pi k} & k \text{ odd} \end{cases}$$

Fourier polynomial of degree 1:

$$f(x) \approx F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x$$

Fourier polynomial of degree 3:

$$f(x) \approx F_3(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x$$



– a better approximation

Higher-degree approximations: (n odd)

$$f(x) \approx F_n(x) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{2}{\pi k}\sin kx$$

Higher-degree Fourier approximations are more accurate.

Continue procedure, letting $n \to \infty$:

<u>Fourier series</u> for the square wave

$$F(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

$$= \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin kx$$

(we don't consider here the important and difficult question of <u>convergence</u> of this infinite series.

In this case, the series converges pointwise to the square wave $f(x)$ at each x except at the points of discontinuity $x = 0, \pm\pi$.)

Fourier series: uses information about f on entire interval (via integrals); good glo<u>bal</u> approximation of function (Taylor series: lo<u>cal</u> approximation)