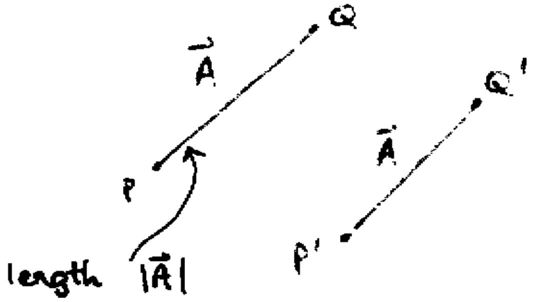
1. Vector Algebra

Vector - has · magnitude
· direction

(alternative notation: $\vec{A}$, $\underline{A}$)



PQ : directed line segment (DLS)

P'Q' : parallel translate of PQ

PQ, P'Q' are <u>equivalent</u> : same magnitude, direction

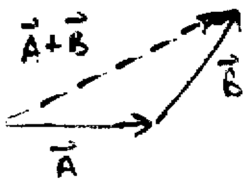
- Vector : a collection of equivalent DLS.
  $\Rightarrow$ PQ, P'Q' : different DLS (different locations) but correspond to same vector

- Zero vector $\vec{0}$: corresponds to degenerate DLS (P = Q)

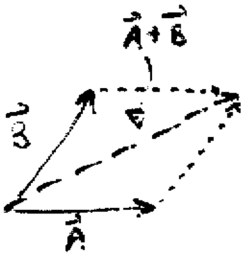
- <u>Magnitude</u> of vector $\vec{A}$ is $|\vec{A}|$: length of PQ.

- $-\vec{A}$: vector with same magnitude as $\vec{A}$, opposite direction corresponds to DLS QP.

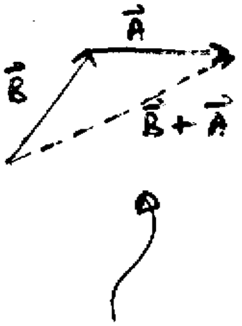
Addition : $\vec{A} + \vec{B}$



Triangle law

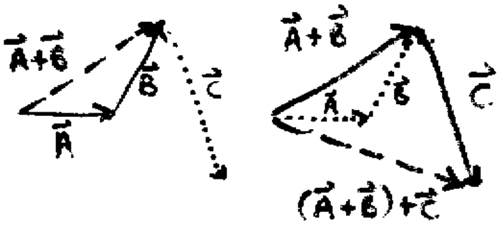


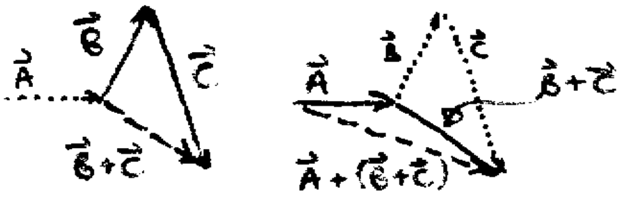
Parallelogram law



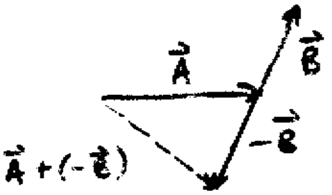
Vector addition is <u>commutative</u> : $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

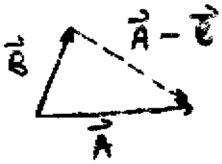
Associative : $(\vec A + \vec B) + \vec C = \vec A + (\vec B + \vec C)$





Subtraction : $\vec A - \vec B = \vec A + (-\vec B)$





must be added to $\vec B$ to give $\vec A$

Also:

$\vec A - \vec A = \vec 0$, $\vec A + \vec 0 = \vec 0 + \vec A = \vec A$

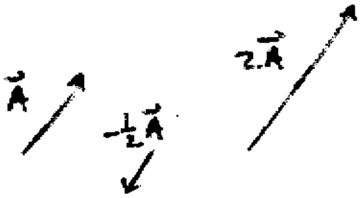
Scalar multiplication :

If s is a scalar (number: $s \in \mathbb{R}$) and $\vec A$ is a vector, $s\vec A$ is a vector :

magnitude $|s\vec A| = |s|\,|\vec A|$ ← $|s|$ times magnitude of $\vec A$

direction : same direction as $\vec A$ if $s > 0$

opposite direction to $\vec A$ if $s < 0$



Properties : • $\underset{\text{scalar}}{0}\,\vec A = \underset{\text{vector}}{\vec 0}$, $1\vec A = \vec A$, $(-1)\vec A = -\vec A$

$s, t \in \mathbb{R}$

• Distributive : $(s+t)\vec A = s\vec A + t\vec A$

$s(\vec A + \vec B) = s\vec A + s\vec B$

Scalar multiplication is commutative, associative → • $(st)\vec A = s(t\vec A) = t(s\vec A) = (ts)\vec A$

Unit vector : a vector with magnitude 1

If $\vec A \neq \vec 0$, the unit vector in the direction of $\vec A$ is $\hat A = \dfrac{\vec A}{|\vec A|}$

denotes unit vector

divide $\vec A$ by its magnitude

Properties of <u>magnitude</u> : ("<u>Norm</u>")

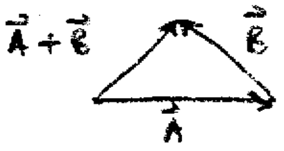
- $|\vec{A}| \geq 0$ for all vectors $\vec{A}$

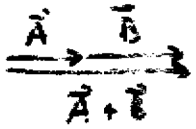
  $|\vec{A}| = 0$ iff $\vec{A} = \vec{0}$

  ↖ "if and only if"

- Triangle inequality : $|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$

The length of one side of a triangle cannot exceed the sum of the lengths of the other two sides.





Vector Space

The above concepts may be extended to more general classes of mathematical objects :

Vector space : a set V of elements $\overset{\text{"vectors"}}{u, v, \ldots} \in V$

with an associated set of scalars

(usually $\mathbb{R}$: real numbers or $\mathbb{C}$: complex numbers)

so that <u>addition</u> and <u>scalar multiplication</u> are defined, and satisfy the above properties (commutativity, associativity, distributivity)

ie (for a vector space V over the reals $\mathbb{R}$)

if $u, v \in V$, $\quad s, t \in \mathbb{R}$

then $u + v \in V$, $\quad su \in V$

(in general $\underbrace{su + tv}_{} \in V$)

all linear combinations of elements of V are also in V

and $u + v = v + u$, $\quad s(u+v) = su + sv$, etc.

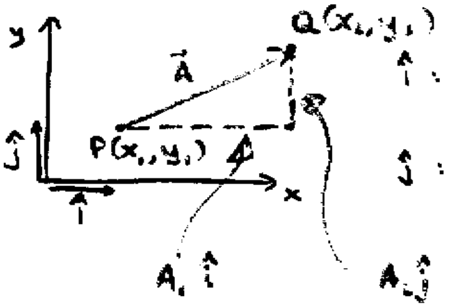
Example: $V = \{\text{quadratic polynomials}\} = \{a + bx + cx^2,\ a, b, c \in \mathbb{R}\}$

So far our treatment has been purely geometric, without reference to any coordinate system.

Cartesian coordinates

Plane :

- introduce perpendicular x, y axes



$\hat{i}$: unit vector parallel to x-axis

$\hat{j}$: unit vector in (positive) y-direction

Every vector in the plane can be written uniquely as

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j}$$

(every vector in the plane is a linear combination of $\hat{i}$ and $\hat{j}$; equivalently, $\hat{i}, \hat{j}$ form a basis for the set of planar vectors

A_1 , A_2 : components of $\vec{A}$ in x, y directions.

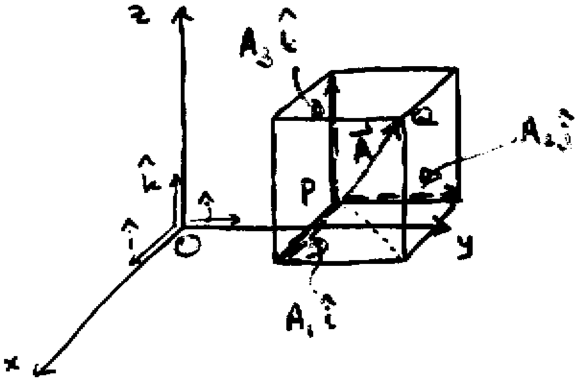
Magnitude : by Pythagorean theorem: $|\vec{A}| = \sqrt{A_1^2 + A_2^2}$
(length)

If $\vec{A}$ is represented by the DLS PQ joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then

$$\vec{A} = \underbrace{(x_2 - x_1)}_{A_1} \hat{i} + \underbrace{(y_2 - y_1)}_{A_2} \hat{j} \qquad |\vec{A}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Three-dimensional Space

- 3 mutually perpendicular x, y, z-axes
right-handed coordinate system
unit vectors $\hat{i}, \hat{j}, \hat{k}$



Unique expansion : $\vec{A} = A_1 \hat{\imath} + A_2 \hat{\jmath} + A_3 \hat{k}$

A_1, A_2, A_3 : components of $\vec{A}$

A_1 : component of $\vec{A}$ in x-direction (direction of $\hat{\imath}$)

– orthogonal projection of the vector $\vec{A}$ in the x-direction

(similarly for A_2, A_3)

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

If PQ is the DLS representing A for points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ then

$$\vec{A} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

(note: $\vec{A}$ = OQ − OP, where O (0,0,0) is the (arbitrary) origin of the coordinate system)

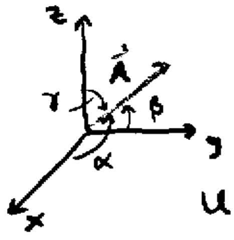
Vector addition, scalar multiplication proceed componentwise

$\vec{A}, \vec{B}$ vectors, scalar $s \in \mathbb{R}$: eg $s\vec{A} + \vec{B} = (sA_1 + B_1)\hat{\imath} + (sA_2 + B_2)\hat{\jmath} + (sA_3 + B_3)\hat{k}$

Examples: $\vec{A} = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$, $\vec{B} = \hat{\imath} + 2\hat{\jmath}$

- $\vec{A} + \vec{B} = 4\hat{\imath} + 4\hat{\jmath} - \hat{k}$
- $-2\vec{A} = -6\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$
- $|\vec{A}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$
- unit vector in direction of $\vec{B}$: $\hat{B} = \frac{1}{|\vec{B}|}(\hat{\imath} + 2\hat{\jmath}) = \frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{\jmath}$

Alternatively, we can describe a vector by giving its magnitude $|\vec{A}|$ and direction, for instance giving the angles α, β, γ between $\vec{A}$ and the positive x-, y-, z-axes, respectively.



Usually prescribe direction cosines :

$$\cos\alpha = \frac{A_1}{|\vec{A}|}, \quad \cos\beta = \frac{A_2}{|\vec{A}|}, \quad \cos\gamma = \frac{A_3}{|\vec{A}|}$$

Direction cosines: $\cos\alpha = \frac{A_1}{|\vec{A}|}$, $\cos\beta = \frac{A_2}{|\vec{A}|}$, $\cos\gamma = \frac{A_3}{|\vec{A}|}$

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Note:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{A_1^2 + A_2^2 + A_3^2}{|\vec{A}|^2} = 1$$

$\Rightarrow \alpha, \beta, \gamma$ cannot be chosen arbitraily.

eg Give all unit vectors with $\cos\alpha = \frac{1}{2}$, $\cos\beta = \frac{1}{2}$:

$$A_1 = \overset{1}{|\vec{A}|}\cos\alpha = 1.\tfrac{1}{2} = \tfrac{1}{2}, \quad A_2 = |\vec{A}|\cos\beta = \tfrac{1}{2}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \tfrac{1}{4} + \tfrac{1}{4} + \cos^2\gamma = 1 \Rightarrow \cos^2\gamma = \tfrac{1}{2} \Rightarrow \cos\gamma = \pm\tfrac{1}{\sqrt{2}}$$

$$\Rightarrow A_3 = |\vec{A}|\cos\gamma = \pm\frac{1}{\sqrt{2}}$$

$\Rightarrow$ there are two unit vectors, $\hat{A} = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} \pm \frac{1}{\sqrt{2}}\hat{k}$.

As in this example, we can compute the components of a vector (w.r.t. a particular othagonal coordinate system) from its geometic characteristics, the magnitude and the angles (or direction cosines) to the axes.

"with respect to"

Types of vectors:

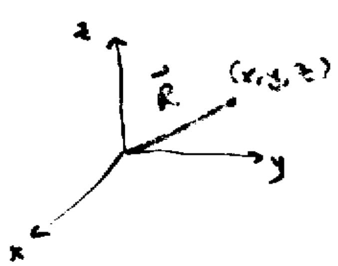
In many problems we wish to consider position eg of a particle:

- Choose a particular coordinate system (locations of x,y,z axes)

-then the position vector is represented by the DLS from the origin (0,0,0) to the position (x,y,z) of the particle.

Position vector $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

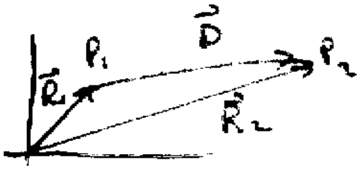
(depends on choice of coordinate system, of origin)



- If the particle moves from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$, the displacement vector is represented by the DLS P_1P_2:

Displacement vector $\vec{D} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$

$$= \underset{\text{final position vector}}{\vec{R}_2} - \underset{\text{initial position vector}}{\vec{R}_1}$$



(displacement vector does not depend on the coordinate system — although its components do — or on origin).

This discussion of vectors: · geometric, basis-independent
"physics" · components are given explicitly with the basis

Alternative: one can define a vector as an ordered n-tuple;
"mathematics" an element of $\mathbb{R}^n$.

Most important: $n = 2$ $\mathbb{R}^2$: x-y plane (Cartesian plane)
$n = 3$ $\mathbb{R}^3$: 3-dimensional space

- this immediately fixes a basis and representation.

Easy to generalize to higher dimensions:

Vector: an element of $\mathbb{R}^n$ ordered list of n real numbers

$$\vec{x} = (x_1, x_2, x_3, \dots, x_n)$$

Addition, scalar multiplication are defined componentwise:

$\vec{x}, \vec{y} \in \mathbb{R}^n$
$t \in \mathbb{R}$

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$
$$t\vec{x} = \langle tx_1, tx_2, \dots, tx_n \rangle$$

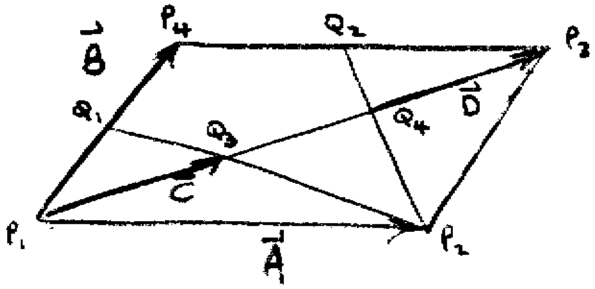
Then we can define (see later)

Scalar field: $f : \mathbb{R}^n \to \mathbb{R}$
Vector field: $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$

We can then interpret elements of $\mathbb{R}^n$ (esp $\mathbb{R}^2, \mathbb{R}^3$) geometrically - but: this can obscure the geometry, hinder the use of other bases.

Geometry using vectors:

eg Given line segments from the vertex of a parallelogram to the midpoints of opposite sides – show that they trisect a diagonal



Write the DLS in terms of corresponding vectors

eg the DLS P_1P_3 (diagonal) is given by $\vec{A}+\vec{B}$

P_1Q_1 : $\frac{1}{2}\vec{B}$

P_2Q_1 : $\frac{1}{2}\vec{B}-\vec{A}$

P_2Q_3 : $\vec{C}-\vec{A}$

- $\vec{C}$ lies on the diagonal

$$\Rightarrow \vec{C} = s(\vec{A}+\vec{B}) \quad \text{for some scalar } s$$

–we wish to show $s = 1/3$.

- Since the tip of $\vec{C}$ (at Q_3) lies on the line segment Q_1P_2:

$$\vec{C}-\vec{A} = t\left(\tfrac{1}{2}\vec{B}-\vec{A}\right) \quad \text{for some scalar } t$$

$$\Rightarrow \vec{C} = \vec{A} + t\left(\tfrac{1}{2}\vec{B}-\vec{A}\right)$$

- Equating the expressions for $\vec{C}$:

$$s(\vec{A}+\vec{B}) = (1-t)\vec{A} + \tfrac{1}{2}t\,\vec{B}$$

$$\Rightarrow (s+t-1)\vec{A} = \left(\tfrac{1}{2}t - s\right)\vec{B}$$

- $\vec{A}$ and $\vec{B}$ are independent (not parallel)

$$\Rightarrow \text{this can hold only if} \quad \left.\begin{array}{l} s+t-1=0 \\ \tfrac{1}{2}t - s = 0 \end{array}\right\}$$

Substitute $s = \frac{1}{2}t$ (i.e. $t = 2s$) into 1st equation:

$$3s - 1 = 0 \Rightarrow \underline{s = 1/3}$$

Similarly, $\vec{D} = \frac{1}{3}(\vec{A}+\vec{B})$ (same derivation, or symmetry)

Scalar Product

Definition: Scalar product of vectors $\vec{A}, \vec{B}$:

a scalar/number →
$$\boxed{\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos \theta}$$

Where θ is the angle between the vectors $\vec{A}, \vec{B}$.

Names: Scalar product, inner product, dot product

Properties from this definition:

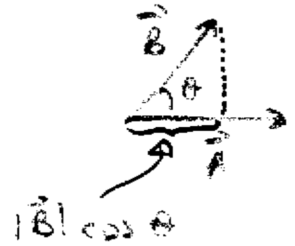
- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- if $\vec{A} = \vec{0}$ or $\vec{B} = \vec{0}$, then $\vec{A} \cdot \vec{B} = 0$
  (θ undefined in this case)
- if $\vec{A}$ and $\vec{B}$ are perpendicular (orthogonal)
  then $\theta = \pi/2 \Rightarrow \cos\theta = 0 \Rightarrow \vec{A} \cdot \vec{B} = 0$.

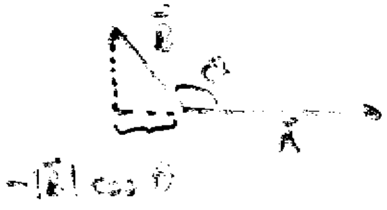
Orthogonality condition for nonzero $\vec{A}, \vec{B}$: $\boxed{\vec{A} \perp \vec{B} \Leftrightarrow \vec{A} \cdot \vec{B} = 0}$

- if $\vec{B} = \vec{A}$, then $\theta = 0 \Rightarrow \cos\theta = 1$
  $\Rightarrow \vec{A} \cdot \vec{A} = |\vec{A}|^2 \cdot 1 = |\vec{A}|^2$

Norm from inner product (Magnitude): $\boxed{|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}}$

Geometric interpretation:





$$\vec{A} \cdot \vec{B} = \underbrace{(\text{length}}_{|\vec{A}|} \text{ of } \vec{A})(\text{signed } \underbrace{\text{component of } \vec{B}}_{\text{orthog. projection of } \vec{B} \text{ onto } \vec{A}} \text{ along } \vec{A})$$

Symmetric in $\vec{A}, \vec{B}$ ↘ $= (\text{length of } \vec{B})(\text{signed component of } \vec{A} \text{ along } \vec{B})$

Component form of scalar product:

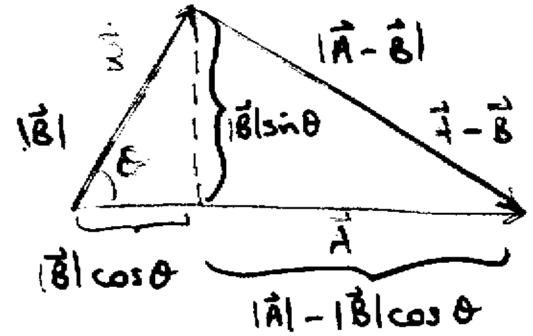
($\{A_i\}$, $\{B_i\}$ are components of $\vec{A}$, $\vec{B}$ w.r.t. a given orthonormal basis) : with respect to

$$\boxed{\vec{A}\cdot\vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3} = \sum_{i=1}^{3} A_i B_i$$

Proof.

- If either $\vec{A}$ or $\vec{B}$ is $\vec{0}$, both sides are 0
- If $\vec{B} = t\vec{A}$ ($\vec{B}$ parallel to $\vec{A}$), l.h.s. = r.h.s. = $t|\vec{A}|^2$
- $\vec{A}, \vec{B}$ not parallel:

lengths:



Obtain two expressions for $|\vec{A}-\vec{B}|^2$

From components:

$$\begin{aligned} |\vec{A}-\vec{B}|^2 &= (A_1-B_1)^2 + (A_2-B_2)^2 + (A_3-B_3)^2 \\ &= A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - 2(A_1B_1 + A_2B_2 + A_3B_3) \\ &= |\vec{A}|^2 + |\vec{B}|^2 - 2(A_1B_1 + A_2B_2 + A_3B_3) \end{aligned}$$

From geometry:

$$\begin{aligned} |\vec{A}-\vec{B}|^2 &= (|\vec{B}|\sin\theta)^2 + (|\vec{A}| - |\vec{B}|\cos\theta)^2 \\ &= |B|^2\sin^2\theta + |\vec{A}|^2 + |\vec{B}|^2\cos^2\theta - 2|\vec{A}|\,|\vec{B}|\cos\theta \\ &= |\vec{A}|^2 + |\vec{B}|^2 - 2\underbrace{|\vec{A}|\,|\vec{B}|\cos\theta}_{\vec{A}\cdot\vec{B}} \qquad \text{(Law of Cosines)} \end{aligned}$$

Comparing these expressions,

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

⌐Alternative approach - more readily generalized:

Define $\vec{A}\cdot\vec{B}$ as $A_1B_1 + A_2B_2 + A_3B_3$, then use the above argument to show $\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$ ⌟

Scalar product $\vec{A}\cdot\vec{B} = \sum_{i=1}^{n} A_i B_i$ (in n dimensions)

Properties :

- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

  $(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C} + \vec{B}\cdot\vec{C}$

  $s(\vec{A}\cdot\vec{C}) = (s\vec{A})\cdot\vec{C} = \vec{A}\cdot(s\vec{C})$ $\quad s\in\mathbb{R}$
- $\vec{A}\cdot\vec{A} \geq 0$, $\vec{A}\cdot\vec{A} = 0$ iff $\vec{A} = \vec{0}$

General properties of (real) inner product

Magnitude (norm) in terms of inner product: $|\vec{A}|^2 = \vec{A}\cdot\vec{A}$

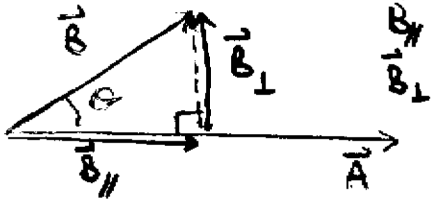
Angle between two vectors : $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$

Cauchy-Schwarz inequality: $|\vec{A}\cdot\vec{B}| \leq |\vec{A}|\,|\vec{B}|$

(since $|\cos\theta| \leq 1$)

Projection :

Decomposition into parallel, perpendicular components



$\vec{B}_{\parallel}$ parallel to $\vec{A}$

$\vec{B}_{\perp}$ perpendicular to $\vec{A}$

(Signed) length of $\vec{B}_{\parallel}$: $|\vec{B}|\cos\theta = \boxed{\dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|}} = \vec{B}\cdot\hat{A}$

(positive if angle is acute, $\theta < \pi/2$)

$\text{comp}_{\vec{A}}\vec{B}$ = component of $\vec{B}$ along $\vec{A}$

↑unit vector $\vec{A}/|\vec{A}|$

$$\Rightarrow \quad \vec{B}_{\parallel} = \left(\frac{\vec{B}\cdot\vec{A}}{|\vec{A}|}\right)\frac{\vec{A}}{|\vec{A}|} = (\vec{B}\cdot\hat{A})\hat{A} = \left(\frac{\vec{B}\cdot\vec{A}}{\underbrace{\vec{A}\cdot\vec{A}}_{|\vec{A}|^2}}\right)\vec{A}$$

$\text{proj}_{\vec{A}}\vec{B}$ = Projection of $\vec{B}$ onto $\vec{A}$

Decomposition : $\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$

$$\vec{B}_{\perp} = \vec{B} - \vec{B}_{\parallel} = \vec{B} - \frac{\vec{B}\cdot\vec{A}}{|\vec{A}|^2}\vec{A}$$

(Note: $\vec{B}\cdot\vec{A} = \vec{B}_{\parallel}\cdot\vec{A}$

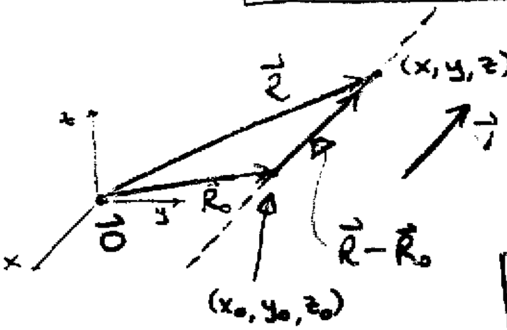
eg component of $\vec{B} = 8\hat{\imath} + \hat{\jmath}$ in the direction of $\vec{A} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ is $|\vec{B}_{\parallel}| = \dfrac{\langle 8,1,0\rangle\cdot\langle 1,2,-2\rangle}{\sqrt{1^2+2^2+2^2}}$

Equations of lines and planes

Equation of line through the point (x_0, y_0, z_0) with position vector $\vec{R}_0$, in direction of vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, is

$$\boxed{\vec{R} = \vec{R}_0 + t\vec{v}} \qquad (t \in \mathbb{R})$$

← parametric form



(x,y,z), position vector $\vec{R}$: any point on line

$[\vec{R} - \vec{R}_0$ is parallel to $\vec{v}]$

Equivalent:

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\}$$

t : parameter (not unique)

Eliminate t :

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \qquad (= t)$$

← non-parametric form

(can read off components $\langle a,b,c\rangle$ of $\vec{v}$, coordinates (x_0, y_0, z_0) of $\vec{R}_0$)

Line: one-dimensional (subspace)
- need a point on the line and one vector parallel to line ↗$\vec{v}$

(one element in basis)

Plane: two-dimensional
- need a point in the plane, and two vectors $\vec{A}$ and $\vec{B}$ (linearly independent i.e. not parallel)

(two elements in basis : all vectors in plane can be written as linear combinations of $\vec{A}, \vec{B}$)

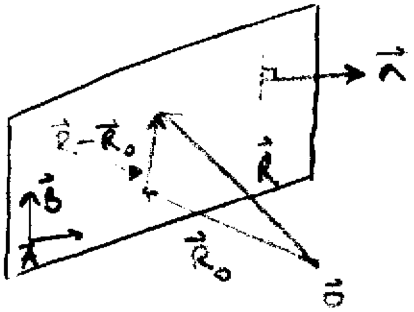
Plane : $\vec{R}_0$ · position vector of point (x_0, y_0, z_0) in plane

$\vec{R}$ - any other point (x, y, z)

Then

$$\boxed{\vec{R} - \vec{R}_0 = s\vec{A} + t\vec{B}} \qquad (s, t \in \mathbb{R})$$

parametric form



$\vec{R} - \vec{R}_0$: arbitrary vector in plane

Vectors, $\vec{A}$, $\vec{B}$ span plane

Alternative : Normal vector $\vec{n}$

For a plane (2-dim) in $\mathbb{R}^3$, orthogonal subspace is $3-2=1$-di ie $\vec{n}$ unique (up to magnitude, sign)

$\vec{R} - \vec{R}_0$ is orthogonal to $\vec{n}$

so $(\vec{R} - \vec{R}_0) \cdot \vec{n} = 0 \quad \Rightarrow \quad \vec{R} \cdot \vec{n} = d \quad$ where $d = \vec{R}_0 \cdot \vec{n}$

Equivalent : $(x - x_0)a + (y - y_0)b + (z - z_0)c = 0$

$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\Rightarrow \boxed{ax + by + cz = d} \qquad \text{where } d = ax_0 + by_0 + cz_0$$

non-parametric form

(can read off components $\langle a, b, c \rangle$ of normal vector to plane)

Note: a line is the intersection of two (non-parallel) planes.

---

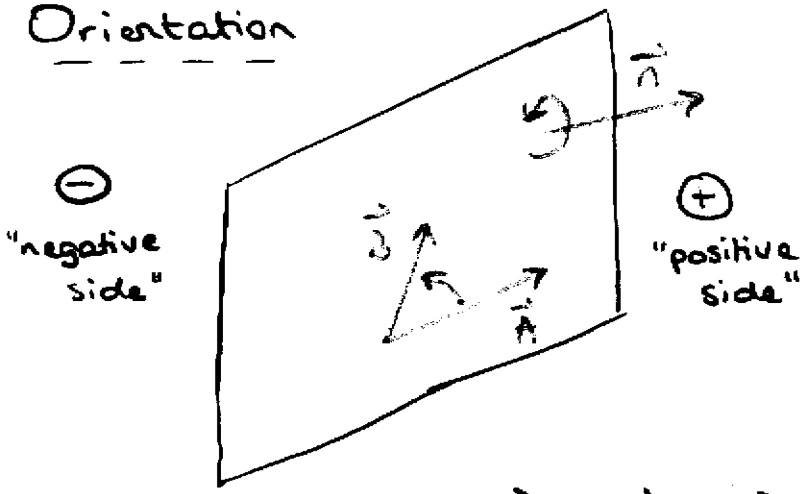
eg Distance between arbitrary point $\vec{R}_1$: (x_1, y_1, z_1) and plane $ax + by + cz = d$ (normal $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$) :

$\vec{R}_0$: any position vector in plane

Distance : absolute value of component of $\vec{R}_1 - \vec{R}_0$ ⊥ to plane ie ∥ to $\vec{n}$

$$\Rightarrow \text{distance} = \frac{|(\vec{R}_1 - \vec{R}_0) \cdot \vec{n}|}{|\vec{n}|} = \frac{|\vec{R}_1 \cdot \vec{n} - d|}{|\vec{n}|} = \frac{|ax_1 + by_1 + cz_1 - d|}{(a^2 + b^2 + c^2)^{1/2}}$$

Orientation



$\vec A, \vec B$: ordered pair of non parallel vectors in plane

Rotation from $\vec A$ to $\vec B$ (along smallest angle) is the "positive sense" of rotation.

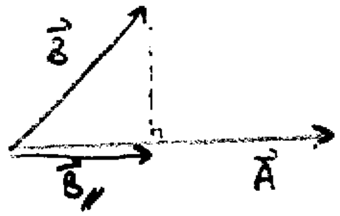
$\Rightarrow$ a plane is oriented by giving the vectors $\vec A$, $\vec B$ (in order).

Right-hand rule: If fingers of right hand curl in positive sense of rotation, the thumb points to the positive side of the plane.

Vectors, Matrices and Projections

[In this section, we consider three-dimensional vectors $\in \mathbb{R}^3$ —all arguments generalize to $\mathbb{R}^n$]

Projection:



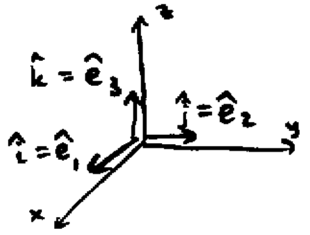
$$\vec B_{\parallel} = \frac{\vec B \cdot \vec A}{|\vec A|^2}\,\vec A = \underbrace{(\vec B \cdot \hat A)}_{\text{comp}_{\vec A}\vec B}\,\hat A = \text{proj}_{\vec A}\,\vec B$$

$\vec A$, $\vec B$ orthogonal : $\vec A \cdot \vec B = 0$

Define
$\hat e_1 = \hat\imath$, $\hat e_2 = \hat\jmath$, $\hat e_3 = \hat k$

Fix a Cartesian coordinate system with unit vectors

$$\{\hat\imath, \hat\jmath, \hat k\} = \{\hat e_1, \hat e_2, \hat e_3\}$$



Then $\hat\imath \cdot \hat\imath = \hat\jmath \cdot \hat\jmath = \hat k \cdot \hat k = 1$
$\hat\imath \cdot \hat\jmath = \hat\imath \cdot \hat k = \hat\jmath \cdot \hat k = 0$

[ie $\hat e_i \cdot \hat e_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$, $i,j = 1,2,3$]

The vectors $\{\hat\imath, \hat\jmath, \hat k\} = \{\hat e_1, \hat e_2, \hat e_3\}$ form an <u>orthonormal</u> basis for the space of all three-dimensional vectors.

We can expand any vector $\vec A$ in terms of this basis, and find the components by taking inner products with the basis vectors:

$$\vec A = A_1\hat\imath + A_2\hat\jmath + A_3\hat k$$

$$\Rightarrow \vec A\cdot\hat\imath = A_1\underbrace{\hat\imath\cdot\hat\imath}_{=1} + A_2\underbrace{\hat\jmath\cdot\hat\imath}_{=0} + A_3\underbrace{\hat k\cdot\hat\imath}_{=0} = A_1 = \text{proj}_{\hat\imath}\vec A$$

$$= |\vec A|\cos\alpha$$

(recall: direction cosine $\cos\alpha = \frac{A_1}{|\vec A|} = \frac{\vec A\cdot\hat\imath}{|\vec A|} = \hat A\cdot\hat\imath$)

Similarly $\vec A\cdot\hat\jmath = A_2 = |\vec A|\cos\beta$, $\quad \vec A\cdot\hat k = A_3 = |\vec A|\cos\gamma$

Thus we can write

$$\vec A = \underbrace{(\vec A\cdot\hat\imath)}_{A_1}\hat\imath + \underbrace{(\vec A\cdot\hat\jmath)}_{A_2}\hat\jmath + \underbrace{(\vec A\cdot\hat k)}_{A_3}\hat k$$

$$= \sum_{i=1}^{3}(\vec A\cdot\hat e_i)\,\hat e_i$$

⌈Look at pages 20-21: Vectors and Matrices here⌋

Now let $\{\vec b_1, \vec b_2, \vec b_3\}$ be any set of mutually orthogonal vectors $\quad (\vec b_i\cdot\vec b_j = 0 \text{ for } i \neq j)$

$\Rightarrow$ they form an independent set, hence a basis

$\Rightarrow$ we can write (for any $\vec A$)

$$\vec A = t_1\vec b_1 + t_2\vec b_2 + t_3\vec b_3 \qquad \begin{pmatrix}\text{3 equations in}\\ \text{3 unknowns}\end{pmatrix}$$

Since the $\{\vec b_i\}$ are <u>orthogonal</u>, we can easily find the coefficients $\{t_i\}$:

$$\vec A\cdot\vec b_1 = t_1\,\vec b_1\cdot\vec b_1 + t_2\underbrace{\vec b_2\cdot\vec b_1}_{=0} + t_3\underbrace{\vec b_3\cdot\vec b_1}_{=0} = t_1\,\vec b_1\cdot\vec b_1$$

$$\Rightarrow t_1 = \frac{\vec A\cdot\vec b_1}{\vec b_1\cdot\vec b_1} \qquad \text{Similarly } t_2 = \frac{\vec A\cdot\vec b_2}{\vec b_2\cdot\vec b_2}, \quad t_3 = \frac{\vec A\cdot\vec b_3}{\vec b_3\cdot\vec b_3}$$

and $$\vec A = \sum_{i=1}^{3}\left(\frac{\vec A\cdot\vec b_i}{\vec b_i\cdot\vec b_i}\right)\vec b_i = \text{proj}_{\vec b_1}\vec A + \text{proj}_{\vec b_2}\vec A + \text{proj}_{\vec b_2}\vec A$$

From an orthogonal basis, we can construct an orthonormal basis (3 mutually orthogonal unit vectors)

by $\underset{\hat{e}_i' \nearrow}{\hat{b}_i} = \frac{\vec{b}_i}{|\vec{b}_i|}$, $i = 1,2,3$ $\left(\hat{b}_i \cdot \hat{b}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \equiv \delta_{ij} \right)$

Then $\{ \hat{b}_1, \hat{b}_2, \hat{b}_3 \}$ define a new Cartesian coordinate system

(write $\{ \hat{e}_1', \hat{e}_2', \hat{e}_3' \} = \{ \hat{\imath}', \hat{\jmath}', \hat{k}' \}$ for the basis vectors)

Change of coordinates: Linear Orthogonal Transformations

Expansion of vector $\vec{A}$ in this new coordinate system:

$$\vec{A} = A_1' \hat{e}_1' + A_2' \hat{e}_2' + A_3' \hat{e}_3' \qquad \hat{e}_i' \cdot \hat{e}_j' = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

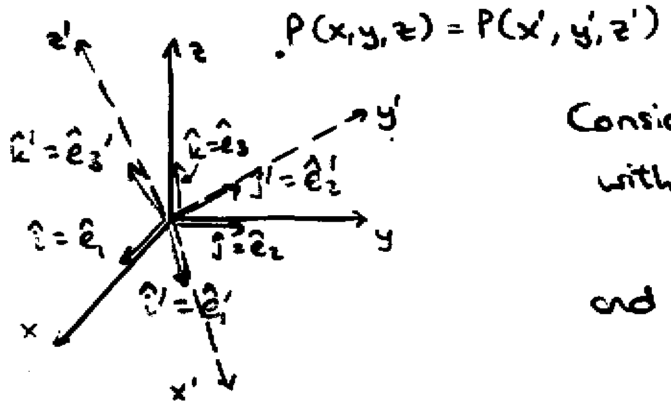
$$\Rightarrow \vec{A} \cdot \hat{e}_1' = A_1' \hat{e}_1' \cdot \hat{e}_1' + 0 + 0 = A_1' \cdot 1 \quad \nwarrow \hat{e}_1' \cdot \hat{e}_1' = 1^2 = 1$$

$$\Rightarrow A_1' = \vec{A} \cdot \hat{e}_1'$$

Similarly $A_2' = \vec{A} \cdot \hat{e}_2'$, $A_3' = \vec{A} \cdot \hat{e}_3'$

How are the components A_j' w.r.t. the new coordinate system related to the old components A_j? "with respect to"

$(j = 1,2,3)$



Consider first a point P, with coordinates (x, y, z) w.r.t. the old axes $\{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \}$, and coordinates (x', y', z') in the new $\{ \hat{e}_1', \hat{e}_2', \hat{e}_3' \}$ system.

(Assume the origin O is invariant under the coordinate change)

Position vector of P: DLS OP.

$$\vec{R} = x \hat{e}_1 + y \hat{e}_2 + z \hat{e}_3$$

$$= x' \hat{e}_1' + y' \hat{e}_2' + z' \hat{e}_3'$$

Take scalar product with old basis vector $\hat{e}_1$:

$$\vec{R}\cdot\hat{e}_1 = x = x'\,\hat{e}_1'\cdot\hat{e}_1 + y'\,\hat{e}_2'\cdot\hat{e}_1 + z'\,\hat{e}_3\cdot\hat{e}_1$$

$$= J_{11}\,x' + J_{12}\,y' + J_{13}\,z' ,$$

where we define

$$J_{ij} = \hat{e}_j'\cdot\hat{e}_i = \hat{e}_i\cdot\hat{e}_j'$$ = cosine of angle between $\hat{e}_i$ and $\hat{e}_j'$

Similarly

$$\vec{R}\cdot\hat{e}_2 = y = \overset{\hat{e}_1'\cdot\hat{e}_2}{J_{21}}\,x' + J_{22}\,y' + J_{23}\,z'$$

$$\vec{R}\cdot\hat{e}_3 = z = J_{31}\,x' + J_{32}\,y' + J_{33}\,z'$$

J_{1j}, J_{2j}, J_{3j} : direction cosines of $\hat{e}_j'$ w.r.t old system

We can summarize this as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}}_{J\ :\ \text{transformation matrix}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = J \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$J = \begin{pmatrix} \hat{i}\cdot\hat{i}' & \hat{i}\cdot\hat{j}' & \hat{i}\cdot\hat{k}' \\ \hat{j}\cdot\hat{i}' & \hat{j}\cdot\hat{j}' & \hat{j}\cdot\hat{k}' \\ \hat{k}\cdot\hat{i}' & \hat{k}\cdot\hat{j}' & \hat{k}\cdot\hat{k}' \end{pmatrix}$$

This gives the old coordinates in terms of the new.

Now take scalar products of the position vector $\vec{R}$ with new basis vectors:

$$\vec{R}\cdot\hat{e}_1' = x' = x\,\hat{e}_1\cdot\hat{e}_1' + y\,\hat{e}_2\cdot\hat{e}_1' + z\,\hat{e}_3\cdot\hat{e}_1'$$

$$= J_{11}\,x + J_{21}\,y + J_{31}\,z$$

(using the above definition of J_{ij} : direction cosines)

and

$$\vec{R}\cdot\hat{e}_2' = y' = J_{12}\,x + J_{22}\,y + J_{32}\,z$$

$$\vec{R}\cdot\hat{e}_3' = z' = J_{13}\,x + J_{23}\,y + J_{33}\,z$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} J_{11} & J_{21} & J_{31} \\ J_{12} & J_{22} & J_{32} \\ J_{13} & J_{23} & J_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = J^{T} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

transpose of J (reflection in diagonal: rows of J^T are columns of J)

Combining the above:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = J\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = J J^T\begin{pmatrix}x\\y\\z\end{pmatrix} = I\begin{pmatrix}x\\y\\z\end{pmatrix}$$

identity matrix $I = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

and similarly

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = J^T\begin{pmatrix}x\\y\\z\end{pmatrix} = J^T J\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = I\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$$

$$\Rightarrow \quad J^T J = I = J J^T \quad \Rightarrow \quad \boxed{J^T = J^{-1}}$$

ie J is an *orthogonal matrix*: its transpose equals its inverse

Note also (from the expressions for x, y, z i.t.o.* x', y', z')

*"in terms of"

$$\frac{\partial x}{\partial x'} = \hat{e}_1\cdot\hat{e}_1' = J_{11}, \quad \frac{\partial x}{\partial y'} = \hat{e}_1\cdot\hat{e}_2' = J_{12}, \quad \frac{\partial y}{\partial x'} = J_{21}, \quad \dots$$

$$\Rightarrow \quad J = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{pmatrix} = \frac{\partial(x,y,z)}{\partial(x',y',z')}$$

Jacobian matrix

(matrix of partial derivatives)

We have seen how coordinates transform; what about components of vectors?

$$\vec{A} = \underbrace{A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3}_{\text{expansion of }\vec{A}\text{ in old coordinates}} = \underbrace{A_1'\hat{e}_1' + A_2'\hat{e}_2' + A_3'\hat{e}_3'}_{\text{new}}$$

$$\left.\begin{aligned} A_i &= \vec{A}\cdot\hat{e}_i \\ A_i' &= \vec{A}\cdot\hat{e}_i' \end{aligned}\right\} i = 1\text{-}3$$

$$\begin{aligned} \Rightarrow \vec{A}\cdot\hat{e}_1 = A_1 &= A_1'\,\hat{e}_1'\cdot\hat{e}_1 + A_2'\,\hat{e}_2'\cdot\hat{e}_1 + A_3'\,\hat{e}_3'\cdot\hat{e}_1 \\ &= J_{11}A_1' + J_{12}A_2' + J_{13}A_3' \qquad = \sum_{i=1}^{3} J_{1i}A_i' \end{aligned}$$

Similarly

$$\vec{A}\cdot\hat{e}_2 = A_2 = J_{21}A_1' + J_{22}A_2' + J_{23}A_3'$$

$$\vec{A}\cdot\hat{e}_3 = A_3 = J_{31}A_1' + J_{32}A_2' + J_{33}A_3'$$

Thus we find

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = J \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} \quad \text{and similarly} \quad \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \underset{\overset{\uparrow}{=J^{-1}}}{J^T} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

⇒ components of a vector $\vec{A}$ transform just like the coordinates of a point.

Consequence:

If we define a vector in terms of its components, then the components must transform appropriately under a change of coordinates:

If the vector $\vec{A}$ (in $\mathbb{R}^3$) has components A_1, A_2, A_3 in the Cartesian frame of reference $\{\hat{i}, \hat{j}, \hat{k}\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, then under rotation of the coordinate system to $\{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$, the components become A_1', A_2', A_3' given by $\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = J^T \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, where J is an orthogonal matrix,

$$J_{ij} = \hat{e}_i \cdot \hat{e}_j'$$

Note: A scalar is invariant under a change of coordinates

eg. Show that $\vec{A}\cdot\vec{B}$ is a scalar

ie show $\sum_{i=1}^{3} A_i B_i = \sum_{i=1}^{3} A_i' B_i'$:

In component form

$$\sum_{i=1}^{3} A_i B_i = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} J_{ij} A_j' \right) \left(\sum_{k=1}^{3} J_{ik} B_k' \right)$$

(formula for A_i i.t.o. A_i', B_i i.t.o. B_i')

$$= \sum_{j=1}^{3} \sum_{k=1}^{3} A_j' B_k' \left(\underbrace{\sum_{i=1}^{3} J_{ij} J_{ik}}_{= I_{jk} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases} (= \delta_{jk})} \right) = \sum_{j=1}^{3} A_j' B_j'$$

since $\sum_{i=1}^{3} J_{ij} J_{ik} = \sum_{i=1}^{3} (J^T)_{ji} J_{ik} = (J^T J)_{jk} = I_{jk} = \delta_{jk}$ $\blacksquare$

Corollary: Lengths, angles are preserved under orthonormal coordinate changes.

Vectors and Matrices

(can generalize these results, stated for $\mathbb{R}^3$, to $\mathbb{R}^n$)

Linear transformation $\mathcal{L}$: maps a vector onto another vector, $\vec{C} = \mathcal{L}\vec{A}$

(Linear: $\mathcal{L}(\vec{A}_1 + \vec{A}_2) = \mathcal{L}\vec{A}_1 + \mathcal{L}\vec{A}_2$

$\mathcal{L}(s\vec{A}) = s(\mathcal{L}\vec{A})$, $s \in \mathbb{R}$)

Fix a Cartesian coordinate system $\{\hat{i}, \hat{j}, \hat{k}\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

Then $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$

ie $A_1 = \vec{A}\cdot\hat{e}_1$, $A_2 = \vec{A}\cdot\hat{e}_2$, $A_3 = \vec{A}\cdot\hat{e}_3$ are the components of the vector $\vec{A}$ w.r.t. this coordinate system

Notation: Column vector $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ ← components w.r.t. $\{\hat{i}, \hat{j}, \hat{k}\}$

The linear transformation $\mathcal{L}$ is represented (w.r.t. coordinate system) by a matrix L

Matrix: rectangular array of numbers

components l_{ij}

row index ↗ ↖ column index

$$L = \underbrace{\begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{\substack{3\times 3 \text{ matrix} \\ \text{rows} \quad \text{columns}}}$$

(it is often useful to interpret a 3×3 matrix as consisting of 3 row vectors, or 3 column vectors)

Multiplication of matrices:

$$\underset{m\times r}{P} = \underset{m\times n}{L}\ \underset{n\times r}{R} \quad : \text{components} \quad p_{ik} = \sum_{j=1}^{3} l_{ij}\, r_{jk}$$

= scalar product of i^{th} row of L with k^{th} column of R

Matrix multiplication is associative $L(MR) = (LM)R$

and distributive $\begin{cases} L(M+R) = LM + LR \\ (L+M)R = LR + MR \end{cases}$

but <u>not</u> commutative, in general:

$$\underset{\uparrow}{LR} \neq \underset{\uparrow}{RL}$$

contains scalar product of rows of L with columns of R

rows of R with columns of L

Identity matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $IL = L = LI$ for any L.

with components

$$I_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \equiv \delta_{ij} \quad \text{"Kronecker delta"}$$

If $\underset{\uparrow}{LR} = I$, then also $\underset{\uparrow}{RL} = I$, and we write $R = L^{-1}$.

R is "right inverse" of L "left inverse" <u>inverse</u>

Linear transformation $\vec{C} = \mathcal{L}\vec{A}$:

Components of $\vec{C}$ (w.r.t. basis $\{\hat{i}, \hat{j}, \hat{k}\}$)

are given by $\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = L \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$

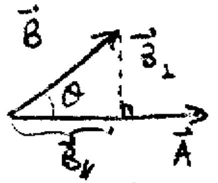
i.e. $C_i = \sum_{j=1}^{3} l_{ij} A_j$

(we can think of $C_i = l_{i1} A_1 + l_{i2} A_2 + l_{i3} A_3 = \begin{pmatrix} l_{i1} & l_{i2} & l_{i3} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$

as the dot product of the i^{th} row of the matrix L and the vector $\vec{A}$).

Note: the new vector $\vec{C}$ is typically in a different direction than $\vec{A}$; if it is in the same direction, $\vec{C} = \lambda \vec{A}$ (ie $\mathcal{L}\vec{A} = \lambda \vec{A}$) then $\vec{A}$ is an <u>eigenvector</u> of L, with eigenvalue λ.

Vector Product



$|\vec{B}_\parallel| = |\vec{B}|\cos\theta$

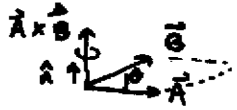
$|\vec{B}_\perp| = |\vec{B}|\sin\theta$

$\vec{B}_\parallel = \left(\vec{A}\cdot\vec{B}/|\vec{A}|\right)\hat{A}$

The vector product of $\vec{A}$ and $\vec{B}$ is defined as

$$\boxed{\vec{A}\times\vec{B} = |\vec{A}|\,|\vec{B}|\sin\theta\ \ \hat{n}}$$

where θ $(0 \le \theta \le \pi)$ is the angle between $\vec{A}$ and $\vec{B}$, and $\hat{n}$ is a unit vector $\perp$ to both $\vec{A}$ and $\vec{B}$, so that $\vec{A}$, $\vec{B}$, $\hat{n}$ form a right-handed system.



Alternative name: cross product : write $\vec{A}\times\vec{B}$ or $\vec{A}\wedge\vec{B}$.

Properties:

- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ due to the rule determining the direction of $\hat{n}$: $\{\vec{A}, \vec{B}, \vec{A}\times\vec{B}\}$ forms a RH system

  cross product is not commutative

- $\vec{A} = \vec{0}$ or $\vec{B} = \vec{0}$ $\Rightarrow$ $\vec{A}\times\vec{B} = \vec{0}$

- $\vec{A}$, $\vec{B}$ parallel vectors $\Rightarrow \theta = 0$ or π $\Rightarrow \sin\theta = 0$

  $\Rightarrow \vec{A}\times\vec{B} = \vec{0}$

  Test for parallelism : $\boxed{\vec{A}\times\vec{B} = \vec{0} \iff \vec{A}, \vec{B} \text{ parallel}}$

  ($\vec{A}$, $\vec{B}$ non zero)

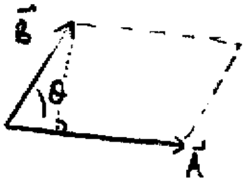
  Special case: for any $\vec{A}$, $\vec{A}\times\vec{A} = \vec{0}$

- Distributive Law : $\vec{A}\times(s\vec{B}+\vec{C}) = s\vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

  (see text, end of §1.12) $(s\vec{A}+\vec{B})\times\vec{C} = s\vec{A}\times\vec{C} + \vec{B}\times\vec{C}$ $s \in \mathbb{R}$

- $|\vec{A}\times\vec{B}| = |\vec{A}|\,|\vec{B}|\sin\theta = |\vec{A}|\,|\vec{B}_\perp|$ = area of parallelogram determined by $\vec{A}$ and $\vec{B}$ ← "base·height"

  = 2 × (area of triangle determined by $\vec{A}$, $\vec{B}$)



Special cases: $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = \vec{0}$

$\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\imath} = \hat{\jmath}$

$\hat{\jmath}\times\hat{\imath} = -\hat{k}$, $\hat{k}\times\hat{\jmath} = -\hat{\imath}$, $\hat{\imath}\times\hat{k} = -\hat{\jmath}$

Component form: (w.r.t. basis $\{\hat{\imath},\hat{\jmath},\hat{k}\}$)

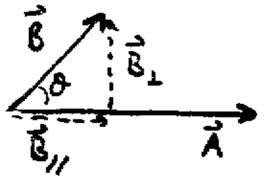
$$\begin{aligned}\vec{A}\times\vec{B} &= (A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}) \times (B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}) \\ &= A_1B_1\,\hat{\imath}\times\hat{\imath} + A_1B_2\,\hat{\imath}\times\hat{\jmath} + \ldots \\ &= A_1B_1\vec{0} + A_1B_2\hat{k} - A_1B_3\hat{\jmath} - A_2B_1\hat{k} + A_2B_2\vec{0} \\ &\quad + A_2B_3\hat{\imath} + A_3B_1\hat{\jmath} - A_3B_2\hat{\imath} + A_3B_3\vec{0}\end{aligned}$$

$$\Rightarrow \boxed{\vec{A}\times\vec{B} = (A_2B_3 - A_3B_2)\hat{\imath} + (A_3B_1 - A_1B_3)\hat{\jmath} + (A_1B_2 - A_2B_1)\hat{k}}$$

$$= \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}\hat{k}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$ symbolic determinant

Parallel - perpendicular decomposition:



$$\vec{B} = \vec{B}_{/\!/} + \vec{B}_\perp$$

$$\vec{B}_{/\!/} = \text{proj}_{\vec{A}}\vec{B} = (\vec{B}\cdot\hat{A})\hat{A} = \left(\frac{\vec{B}\cdot\vec{A}}{\vec{A}\cdot\vec{A}}\right)\vec{A}$$

$$|\vec{B}_\perp| = |\vec{B}|\sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}|}$$

$\vec{B}_\perp$: lies in plane spanned by $\vec{A}, \vec{B}$, $\vec{B}_\perp \perp \vec{A}$

$\vec{A}\times\vec{B}$: normal to plane spanned by $\vec{A}, \vec{B}$ (out of page)

$(\vec{A}\times\vec{B})\times\vec{A}$: normal to $\vec{A}\times\vec{B}$ ie in $\vec{A}-\vec{B}$ plane } $(\vec{A}\times\vec{B})\times\vec{A}$ is in
normal to $\vec{A}$ } direction of $\vec{B}_\perp$
(by RH rule)

$$|(\vec{A}\times\vec{B})\times\vec{A}| = |\vec{A}\times\vec{B}|\,|\vec{A}| = |\vec{A}|^2|\vec{B}|\sin\theta = |\vec{A}|^2|\vec{B}_\perp|$$

Thus $\boxed{\vec{B}_\perp = \dfrac{(\vec{A}\times\vec{B})\times\vec{A}}{\vec{A}\cdot\vec{A}}}$

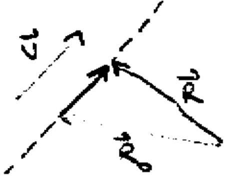
Decomposition:

$$\vec{B} = \frac{\vec{A}\cdot\vec{B}}{\vec{A}\cdot\vec{A}}\vec{A} + \frac{(\vec{A}\times\vec{B})\times\vec{A}}{\vec{A}\cdot\vec{A}} = \vec{B}_{/\!/} + \vec{B}_\perp$$

eg equation of a line: $\vec{v}$: vector parallel to line

$\vec{R}_0$: position vector of a given point on the line

$\vec{R}$: any point on the line



Previously: $\vec{R} = \vec{R}_0 + t\vec{v}$

Equivalent: $\vec{R} - \vec{R}_0$ is parallel to $\vec{v}$

$$\Rightarrow \quad (\vec{R} - \vec{R}_0) \times \vec{v} = \vec{0}$$

$$\Rightarrow \quad \vec{R} \times \vec{v} = \vec{R}_0 \times \vec{v}$$ non-parametric form

eg $\vec{A} = 3\hat{\imath} - \hat{\jmath} + \hat{k}$, $\vec{B} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

Find the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$$

$\Rightarrow |\vec{A} \times \vec{B}| = \sqrt{1 + 16 + 49} = \sqrt{66}$ = area of parallelogram

Find unit vectors ⊥ to both $\vec{A}$ and $\vec{B}$:

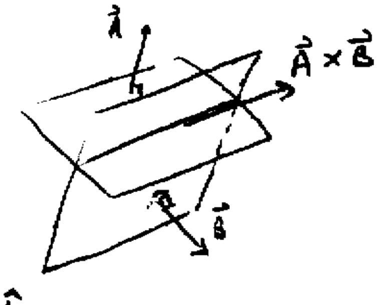
$$\pm \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \pm \frac{1}{\sqrt{66}} \left(-\hat{\imath} + 4\hat{\jmath} + 7\hat{k} \right)$$

eg Find the equation of the line passing through (3, 2, −4) parallel to the line of intersection of the planes

$$x + 3y - 2z = 8 \quad , \quad x - 3y + z = 0$$

Normals to planes: $\vec{A} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ $\quad \vec{B} = \hat{\imath} - 3\hat{\jmath} + \hat{k}$

$\vec{A} \times \vec{B}$ is normal to both $\vec{A}$ and $\vec{B}$, thus it is parallel to both planes



$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ 1 & -3 & 1 \end{vmatrix} = -3\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$$

$\Rightarrow$ line is // to $\hat{\imath} + \hat{\jmath} + 2\hat{k}$.

Line: $\vec{R} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + t\,(\hat{\imath} + \hat{\jmath} + 2\hat{k})$ or $\frac{x-3}{1} = \frac{y-2}{1} = \frac{z+4}{2}$

Interpretation in Mechanics:

Scalar product Force $\vec{F}$ acting through displacement $\vec{D}$:

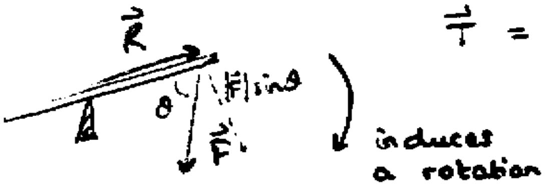
$$\text{Work} = \vec{F} \cdot \vec{D}$$

= product of magnitude of displacement with component of force in direction of displacement

Vector product Torque (relative to the origin) due to force $\vec{F}$ applied at the point with position vector $\vec{R}$:

"moment"

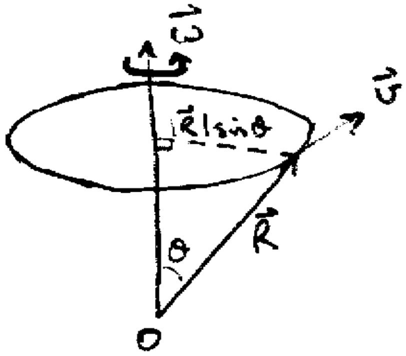
$$\vec{T} = \vec{R} \times \vec{F}$$



direction perpendicular to the plane of $\vec{R}$, $\vec{F}$

Angular velocity: - represented by a vector $\vec{\omega}$, magnitude ω, with direction determined by the right-hand rule:

(if fingers of right hand are wrapped around axis in direction of rotation, the thumb points in the direction of $\vec{\omega}$)



Origin O on axis of rotation

Angular velocity $\vec{\omega}$; $\vec{R}$: position vector of a particle in a rigid body

(Linear) velocity $\vec{v}$ of particle:

$$\vec{v} = \vec{\omega} \times \vec{R}$$

(magnitude $|\vec{v}| = \omega \underbrace{|\vec{R}|\sin\theta}_{\text{radius}}$

direction perpendicular to $\vec{\omega}$, $\vec{R}$)

Magnetic force:

$$\vec{F} = \underset{\text{charge}}{q}\, \underset{\text{velocity}}{\vec{v}} \times \underset{\text{magnetic field}}{\vec{B}}$$

Scalar Triple Product

Definition: $\boxed{[\vec{A}, \vec{B}, \vec{C}] = \vec{A} \cdot (\vec{B} \times \vec{C})}$

$= \vec{A} \cdot \vec{B} \times \vec{C}$ parentheses redundant

Using components:

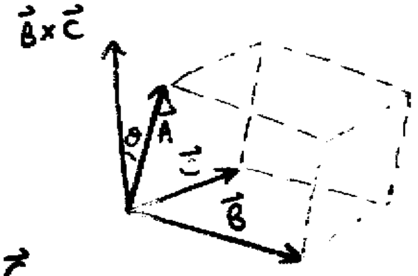
$$[\vec{A}, \vec{B}, \vec{C}] = A_1 (B_2 C_3 - B_3 C_2) + A_2 (B_3 C_1 - B_1 C_3) + A_3 (B_1 C_2 - B_2 C_1)$$

$$\Rightarrow [\vec{A}, \vec{B}, \vec{C}] = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$[\vec{A}, \vec{B}, \vec{C}] > 0$ if $\vec{A}, \vec{B}, \vec{C}$ form a right-handed system

Interpretation: (of scalar triple product and 3×3 determinant)

Parallelepiped, coterminal edges $\vec{A}, \vec{B}, \vec{C}$:



Volume = (area of base) (height)

Area of base parallelogram = $|\vec{B} \times \vec{C}|$

Height : component of $\vec{A}$ ⊥ base ie ∥ $\vec{B} \times \vec{C}$

$$|\text{comp}_{\vec{B} \times \vec{C}} \vec{A}| = |\vec{A}| \, |\cos\theta| = \frac{|\vec{A} \cdot (\vec{B} \times \vec{C})|}{|\vec{B} \times \vec{C}|}$$

$\vec{A}, \vec{B}, \vec{C}$ coplanar ⇒ $\vec{A} \perp \vec{B} \times \vec{C}$ ⇒ Volume = 0

⇒ volume of parallelepiped:

$$= |\vec{B} \times \vec{C}| \, |\vec{A}| |\cos\theta| = |\vec{A} \cdot (\vec{B} \times \vec{C})| = |[\vec{A}, \vec{B}, \vec{C}]|$$

Properties:

- Absolute value of scalar triple product does not depend on the order of the vectors; (from volume interpretation)

(properties of determinants) triple product is invariant under cyclic permutations of $\vec{A}, \vec{B}, \vec{C}$; $\vec{A} \to \vec{B} \to \vec{C} \to \vec{A}$

the sign changes if two of the vectors are switched

$$\Rightarrow [\vec{A}, \vec{B}, \vec{C}] = -[\vec{A}, \vec{C}, \vec{B}] = [\vec{C}, \vec{A}, \vec{B}] = -[\vec{C}, \vec{B}, \vec{A}] = [\vec{B}, \vec{C}, \vec{A}] = -[\vec{B}, \vec{A}, \vec{C}]$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{A} \cdot (\vec{C} \times \vec{B}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = -\vec{C} \cdot (\vec{B} \times \vec{A}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$$

- Since $[\vec{A},\vec{B},\vec{C}] = [\vec{C},\vec{A},\vec{B}]$, we have $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

  $$\Rightarrow \quad \vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

  position of dot and cross can be changed

- Scalar triple product is linear in each factor

  eg $[s\vec{A}+\vec{B},\vec{C},\vec{D}] = s[\vec{A},\vec{C},\vec{D}] + [\vec{B},\vec{C},\vec{D}]$,

  ...

- Any two vectors equal $\Rightarrow$ scalar triple product vanishes

  eg $[\vec{A},\vec{A},\vec{B}] = \vec{A}\cdot\underbrace{(\vec{A}\times\vec{B})}_{\perp\vec{A}} = 0$ (volume of ppd $= 0$)

  (or: $[\vec{A},\vec{A},\vec{B}] \underset{\uparrow}{=} -[\vec{A},\vec{A},\vec{B}] \Rightarrow [\vec{A},\vec{A},\vec{B}] = 0$

  switch first 2 terms (for any scalar t, $t=-t$ $\Rightarrow t=0$))

  $$\Rightarrow [\vec{A}+s\vec{B}+t\vec{C},\vec{B},\vec{C}] = [\vec{A},\vec{B},\vec{C}] + s\overset{0}{[\cancel{\vec{B},\vec{B}},\vec{C}]} + t\overset{0}{[\cancel{\vec{C},\vec{B}},\vec{C}]}$$

- $[\hat{i},\hat{j},\hat{k}] = 1$

- Linear independence: (assume no two of $\underbrace{\vec{A},\vec{B},\vec{C}}_{\text{nonzero}}$ are parallel)

  Note: $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0 \Leftrightarrow \vec{A}\perp\vec{B}\times\vec{C}$

  $\Leftrightarrow$ $\vec{A}$ lies in the plane spanned by $\vec{B}$ and $\vec{C}$

  $\Leftrightarrow$ $\vec{A}$, $\vec{B}$, $\vec{C}$ are <u>coplanar</u>

  Equivalently: $[\vec{A},\vec{B},\vec{C}] = \vec{A}\cdot(\vec{B}\times\vec{C}) \neq 0$

  $\Leftrightarrow$ $\vec{A},\vec{B},\vec{C}$ are not coplanar

  $\Leftrightarrow$ None of the vectors can be written as a linear combination of the other two

  $\Leftrightarrow$ No nontrivial linear combination of $\vec{A},\vec{B},\vec{C}$ vanishes: $r\vec{A}+s\vec{B}+t\vec{C} = \vec{0} \Rightarrow r=s=t=0$

  $\Leftrightarrow$ $\vec{A}$, $\vec{B}$, $\vec{C}$ are linearly independent

<u>Test for linear independence</u> $\boxed{[\vec{A},\vec{B},\vec{C}] = \vec{A}\cdot(\vec{B}\times\vec{C}) \neq 0}$

Matrix inverse:

If the 3×3 matrix M has rows $\vec{A}, \vec{B}, \vec{C}$, $M = \begin{pmatrix} \cdots \vec{A} \cdots \\ \cdots \vec{B} \cdots \\ \cdots \vec{C} \cdots \end{pmatrix}$

then M is invertible iff $\det M = [\vec{A}, \vec{B}, \vec{C}] \neq 0$,

in which case the inverse can be written

in terms of its columns as

$$M^{-1} = \frac{1}{[\vec{A}, \vec{B}, \vec{C}]} \begin{pmatrix} \vdots & \vdots & \vdots \\ \vec{B}\times\vec{C} & \vec{C}\times\vec{A} & \vec{A}\times\vec{B} \\ \vdots & \vdots & \vdots \end{pmatrix}$$

Check: $MM^{-1} = I$

eg $\vec{A} = 3\hat{i} + 2\hat{j} - \hat{k}$, $\vec{B} = 4\hat{i} + \hat{k}$, $\vec{C} = -\hat{j} + 2\hat{k}$

$$\begin{aligned}[\vec{A}, \vec{B}, \vec{C}] &= [3\hat{i} + 2\hat{j} - \hat{k}, 4\hat{i} + \hat{k}, -\hat{j} + 2\hat{k}] \\ &= [3\hat{i} + 2\hat{j} - \hat{k}, 4\hat{i}, -\hat{j}] + [3\hat{i} + 2\hat{j} - \hat{k}, 4\hat{i}, 2\hat{k}] \\ &\quad + [3\hat{i} + 2\hat{j} - \hat{k}, \hat{k}, -\hat{j}] + [3\hat{i} + 2\hat{j} \cancel{-\hat{k}}, \hat{k}, 2\hat{k}) \\ &= [-\hat{k}, 4\hat{i}, -\hat{j}] + [2\hat{j}, 4\hat{i}, 2\hat{k}] + [3\hat{i}, \hat{k}, -\hat{j}] \\ &\quad + (\text{terms with repeated vectors}) \\ &= 4 - 16 + 3 = -9\end{aligned}$$

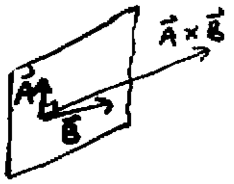
or $[\vec{A}, \vec{B}, \vec{C}] = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3\cdot1 - 2\cdot8 - 1(-4) = -9$

eg equation of a plane:

Vectors $\vec{A}$, $\vec{B}$ span the plane

$\vec{R}_0$: position vector of a given point in the plane

$\vec{R}$: any point in the plane



Previously: $\vec{R} - \vec{R}_0 = s\vec{A} + t\vec{B}$

Equivalent: $\vec{R} - \vec{R}_0$ lies in the plane

$\vec{n} = \vec{A}\times\vec{B}$ is normal to the plane

$\Rightarrow (\vec{R} - \vec{R}_0)\cdot(\vec{A}\times\vec{B}) = 0 \quad \Rightarrow [\vec{R} - \vec{R}_0, \vec{A}, \vec{B}] = 0$

non-parametric equation of plane

Vector Identities

Vector triple product : $\vec{A} \times (\vec{B} \times \vec{C})$

An identity: $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

⌈Motivation : $\vec{B} \times \vec{C}$ is $\perp$ to $\vec{B}, \vec{C}$

$\vec{A} \times (\vec{B} \times \vec{C})$ is $\perp$ $\vec{A}$, and $\perp$ $\vec{B} \times \vec{C}$

$\Rightarrow$ it lies in the plane spanned by $\vec{B}, \vec{C}$

$\Rightarrow$ $\vec{A} \times (\vec{B} \times \vec{C}) = \lambda \vec{B} + \mu \vec{C}$ for some scalars λ, μ

See later for derivation using tensor notation.

All components are products of A_i, B_j, C_k

$\Rightarrow$ $\lambda = \pm \vec{A} \cdot \vec{C}$, $\mu = \pm \vec{A} \cdot \vec{B}$ (with opp. signs, since $\vec{A} \times (\vec{B} \times \vec{C}) = -\vec{A} \times (\vec{C} \times \vec{B})$)

Check signs by trying $\hat{i}, \hat{j}, \hat{k}$:

eg $\hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j} = 0\hat{i} - (\hat{i} \cdot \hat{i})\hat{j}$

$\hat{i} \times (\hat{k} \times \hat{i}) = \hat{i} \times \hat{j} = \hat{k} = (\hat{i} \cdot \hat{i})\hat{k} - 0\,\hat{i}$ ⌋

Similarly: $(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$

– the vector product is <u>not associative</u>

Further identities:

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{A}, \vec{C}, \vec{D}]\vec{B} - [\vec{B}, \vec{C}, \vec{D}]\vec{A}$$

$$= [\vec{D}, \vec{A}, \vec{B}]\vec{C} - [\vec{C}, \vec{A}, \vec{B}]\vec{D}$$

⌈ Let $\vec{u} = \vec{C} \times \vec{D}$: $(\vec{A} \times \vec{B}) \times \vec{u} = [\vec{A} \cdot \vec{u}]\vec{B} - (\vec{B} \cdot \vec{u})\vec{A}$

$= (\vec{A} \cdot (\vec{C} \times \vec{D}))\vec{B} - (\vec{B} \cdot (\vec{C} \times \vec{D}))\vec{A}$ ⌋

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

⌈ $\vec{u} = \vec{C} \times \vec{D}$: $(\vec{A} \times \vec{B}) \cdot \vec{u} = \vec{A} \cdot (\vec{B} \times \vec{u}) = \vec{A} \cdot (\vec{B} \times (\vec{C} \times \vec{D}))$

$= \vec{A} \cdot \left[(\vec{B} \cdot \vec{D})\vec{C} - (\vec{B} \cdot \vec{C})\vec{D}\right]$ ⌋

Tensor Notation

- calculations with vectors are performed componentwise
- use components (and a convenient shorthand) to establish vector identities

(remember that components are relative to a particular basis, and transform appropriately, but a vector identity established in one basis holds in all bases)

$\vec{A}$: vector with components A_1, A_2, A_3 w.r.t a given basis

$\vec{A} = \vec{B} \iff \begin{matrix} A_1 = B_1 \\ A_2 = B_2 \\ A_3 = B_3 \end{matrix}$: write $A_i = B_i$ $\quad \underbrace{(i = 1,2,3)}_{\text{understood}}$

$(\vec{A})_i = (\vec{B})_i$

Ideas and conventions:

- Write equations in component form, using dummy subscripts: $A_i = (\vec{A})_i \leftarrow i^{th}$ component of $\vec{A}$, $i = 1,2,3$

  eg $(\vec{A} + \vec{B})_i = A_i + B_i$ componentwise addition of vectors

- $(i = 1, 2, 3)$ is understood implicitly

- (Einstein) summation convention

  - No index appears more than twice in a single term
  - Whenever a dummy index, say i, appears twice, $\sum_{i=1}^{3}$ is implied : sum over repeated indices (unless explicitly indicated)

  eg $|\vec{A}|^2 = A_1^2 + A_2^2 + A_3^2 = A_i A_i$
  Magnitude (norm) $\quad$ means $\sum_{i=1}^{3} A_i A_i$

  eg Scalar (dot) product :

  $\vec{A} \cdot \vec{B} = A_i B_i$ $\quad$ (means $\sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$)

Examples using tensor notation: Vectors and Tensors

- Coordinate transformations

often written l_{ij}

J : transformation matrix : $J_{ij} = \hat{e}_i \cdot \hat{e}_j'$
$= \cos(\hat{e}_i, \hat{e}_j')$ ← cosine of angle between $\hat{e}_i, \hat{e}_j'$

- J is orthogonal : $J_{ij} J_{ik} = \delta_{ij}$ $\left(\begin{matrix} \text{since } J^T J = I, \text{ and} \\ J_{ij} = (J^T)_{ji} \end{matrix} \right)$

- $\vec{A}$ a vector with components A_i w.r.t. basis $\{\hat{e}_i\}$, A_j' wrt $\{\hat{e}_j'\}$:

$$A_j' = J_{ij} A_i \quad , \qquad A_i = J_{ij} A_j'$$

- A second order Cartesian tensor has 9 components T_{ij}, $i,j = 1,2,3$ in the Cartesian coordinate system $\{\hat{i}, \hat{j}, \hat{k}\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, which under rotation of the frame of reference to $\{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$ become

$$T'_{pq} = J_{ip} J_{jq} T_{ij}$$

Notes:
- A second order tensor (written as $\bar{\bar{T}}$ or T) may be written down as a 3×3 matrix with components T_{ij}.
  In matrix notation the transformation rule is $T' = J^T T J = J^{-1} T J$

- Second order tensors can be identified with linear transformations of the vector space into itself (if $C_i = T_{ij} A_j$, then $C_i' = T_{ij}' A_j'$ ...)

eg 3rd order tensors: $A'_{pqr} = J_{ip} J_{jq} J_{kr} A_{ijk}$

- One can similarly define higher order tensors : an n^{th} order tensor contains 3^n components and transforms by an analogous rule
- δ_{ij}, ϵ_{ijk} are the only isotropic 2nd, 3rd order tensors (components independent of frame of reference)

Important tensors:

- **Kronecker delta**: "substitution tensor"

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For any $\vec{A}$, $\delta_{ij} A_j = A_i$ ← in the sum over subscript j, can drop δ_{ij}, replace j with i

(since in $\sum_{j=1}^{3} \delta_{ij} A_j$, the only nonzero term is when $j = i$.)

eg $\vec{A} \cdot \vec{B} = A_i B_i = A_i \delta_{ij} B_j = \delta_{ij} A_i B_j$ $\left(\overset{\text{means}}{\sum_{i=1}^{3} \sum_{j=1}^{3}} \delta_{ij} A_i B_j \right)$

eg $\delta_{ii} = 3$

- **Permutation tensor**: "permutation symbol / alternator"

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is an even permutation of } (123) \text{ ie } (ijk) = (123), (231), (312) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \text{ ie } (ijk) = (132), (321), (213) \\ 0 & \text{otherwise ie if two or three indices are the same} \end{cases}$$

⌈ Recall:
- permutation of a list/of integers: a rearrangement of the list in another order (for example)
- transposition: two adjacent integers are interchanged eg $(123) \rightarrow (132)$
- $\left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}$ permutation: can be obtained through an $\left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}$ number of transpositions
- cyclic permutations of (123) are (123), (231), (312)

  anticyclic ... (132), (321), (213) ⌋

+1 (cyclic)
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur, bend left] & & 2 \arrow[ll] \end{tikzcd}$$

−1 (anticyclic)
$$\begin{tikzcd} & 1 \arrow[dl, bend right] & \\ 3 \arrow[rr] & & 2 \end{tikzcd}$$

Note: –1. $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$: Subscripts can be permuted cyclically

–2. $\epsilon_{ijk} = -\epsilon_{jik}$: Sign changes if two subscripts are switched

Examples:

· eg Vector (cross) product :

$$\boxed{(\vec{A}\times\vec{B})_i = \epsilon_{ijk} A_j B_k} \longleftarrow \text{means } \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} A_j B_k$$

(eg $\epsilon_{1jk} A_j B_k = A_2 B_3 - A_3 B_2 = (\vec{A}\times\vec{B})_1$

since $\epsilon_{1jk} = 0$ unless $jk = 23$ or 32, $\epsilon_{123} = +1$, $\epsilon_{132} = -1$)

⇒ ϵ_{ijk} is the coefficient of $A_j B_k$ in the i^{th} component of $\vec{A}\times\vec{B}$

· eg Scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i (\vec{B}\times\vec{C})_i = A_i \epsilon_{ijk} B_j C_k$$

$$\Rightarrow \boxed{\vec{A}\cdot\vec{B}\times\vec{C} = [\vec{A},\vec{B},\vec{C}] = \epsilon_{ijk} A_i B_j C_k}$$

$$\uparrow = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} A_i B_j C_k$$

but $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

⇒ ϵ_{ijk} is the coefficient of $A_i B_j C_k$ in the expansion of the determinant

eg show using tensor notation $\vec{A}\cdot(\vec{A}\times\vec{B}) = 0 \leftarrow$ true since $\vec{A} \perp \vec{A}\times\vec{B}$

$$\vec{A}\cdot(\vec{A}\times\vec{B}) = \epsilon_{ijk} A_i A_j B_k = \underset{\uparrow \text{ switch dummy indices } i,j}{\epsilon_{jik} A_j A_i B_k} = \epsilon_{jik} A_i A_j B_k$$

but $\epsilon_{ijk} = -\epsilon_{jik}$, so $\epsilon_{ijk} A_i A_j B_k = -\epsilon_{ijk} A_i A_j B_k$

$$\Rightarrow \vec{A}\cdot(\vec{A}\times\vec{B}) = \epsilon_{ijk} A_i A_j B_k = 0.$$

A useful identity: products of permutation tensors

$$\boxed{\epsilon_{ikm}\,\epsilon_{psm} = \delta_{ip}\,\delta_{ks} - \delta_{is}\,\delta_{kp}} \quad \leftarrow \text{sum over } m$$

$\epsilon_{mik}\,\epsilon_{mps}$

⌈ Proof: RHS = 0 unless it is 1 − 0 or 0 − 1

ie unless <u>either</u> • $i=p,\ k=s$, but $i\neq s$ (or $k\neq p$)

$\Rightarrow$ RHS $= 1-0 = 1$

(try a few cases to convince yourself)

<u>or</u> • $i=s,\ k=p$, but $i\neq p$ (or $k\neq s$)

$\Rightarrow$ RHS $= 0-1 = -1$

If $i=k$ or $p=s$, LHS = RHS = 0.

LHS $\neq 0$ only if m is different from i, k, p and s

So i, k, p, s must be chosen from <u>two</u> of 1, 2, 3

and (at most) one term in the sum over m on LHS is $\neq 0$.

With $i\neq k$, $p\neq s$, there are two possibilities: for LHS $\neq 0$

- $i=p$, $k=s$, $i\neq k$ ($\Rightarrow i\neq s,\ k\neq p$)

  $\Rightarrow \epsilon_{ikm} = \epsilon_{psm} \Rightarrow \text{LHS} = \left\{\begin{matrix}(+1)(+1)\\ \text{or}\\ (-1)(-1)\end{matrix}\right. = +1 = \text{RHS}$

- $i=s$, $k=p$, $i\neq k$ ($\Rightarrow i\neq p,\ k\neq s$)

  $\Rightarrow \epsilon_{ikm} = \epsilon_{spm} = -\epsilon_{psm} \Rightarrow \text{LHS} = -1 = \text{RHS}$

In each case the identity holds. ⌋

eg Vector triple product

Show $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$:

$$\begin{aligned}
[\vec{A}\times(\vec{B}\times\vec{C})]_i &= \epsilon_{ijk}\,A_j\,(\vec{B}\times\vec{C})_k \\
&= \epsilon_{ijk}\,A_j\,\epsilon_{k\ell m}\,B_\ell\,C_m = \epsilon_{ijk}\,\epsilon_{\ell mk}\,A_j\,B_\ell\,C_m \\
&= (\delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell})\,A_j\,B_\ell\,C_m \\
&= A_j\,B_i\,C_j - A_j\,B_j\,C_i \\
&= (\vec{A}\cdot\vec{C})B_i - (\vec{A}\cdot\vec{B})C_i
\end{aligned}$$