Vector Functions of a Single Variable

Definition: Vector-valued function $\vec{F}(t)$:

assigns a vector $\vec{F}$ to each t in some interval $t_1 \le t \le t_2$

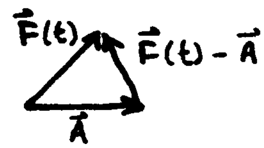
$$\vec{F}(t) = F_1(t)\,\hat{\imath} + F_2(t)\,\hat{\jmath} + F_3(t)\,\hat{k}$$

Limit: $\lim_{t \to t_0} \vec{F}(t) = \vec{A}$

if $\forall \varepsilon > 0 \quad \exists \delta > 0$

s.t. $0 < |t - t_0| < \delta \quad \Rightarrow \quad |\vec{F}(t) - \vec{A}| < \varepsilon$

(ie $|\vec{F}(t)| \to |\vec{A}|$, and angle between $\vec{F}(t)$ and $\vec{A}$ approaches zero.



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Equivalent :

Components of $\vec{F}(t)$ approach components of $\vec{A}$ $\quad \lim_{t \to t_0} F_i(t) = A_i \quad i = 1,2,3$

Continuity: $\vec{F}(t)$ is continuous at t_0 if

$$\lim_{t \to t_0} \vec{F}(t) = \vec{F}(t_0) \quad \Leftrightarrow \quad \lim_{t \to t_0} F_i(t) = F_i(t_0)$$

Derivative: $\vec{F}(t)$ is differentiable at t_0 if the Limit

$$\lim_{\Delta t \to 0} \frac{\vec{F}(t_0 + \Delta t) - \vec{F}(t_0)}{\Delta t} = \vec{F}'(t_0) = \frac{d\vec{F}}{dt}(t_0)$$

exists $\quad \Leftrightarrow F_i'(t)$ exists, $i = 1,2,3$

Note: for a vector written in terms of its components w.r.t a basis:

Differentiate componentwise :

$$\vec{F}'(t) = F_1'(t)\,\hat{\imath} + F_2'(t)\,\hat{\jmath} + F_3'(t)\,\hat{k} \quad : \quad \boxed{\left(\frac{d\vec{F}}{dt}\right)_i = \frac{dF_i}{dt}}$$

Differentiation Rules:

Theorem: If $\vec{F}$ and $\vec{G}$ are differentiable vector functions and s is a differentiable scalar function, then

sum rule 1. $\frac{d}{dt}(\vec{F}+\vec{G}) = \frac{d\vec{F}}{dt} + \frac{d\vec{G}}{dt}$

product rules

2. $\frac{d}{dt}(s\vec{F}) = \frac{ds}{dt}\vec{F} + s\frac{d\vec{F}}{dt}$

3. $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \frac{d\vec{F}}{dt}\cdot\vec{G} + \vec{F}\cdot\frac{d\vec{G}}{dt}$

4. $\frac{d}{dt}(\vec{F}\times\vec{G}) = \frac{d\vec{F}}{dt}\times\vec{G} + \vec{F}\times\frac{d\vec{G}}{dt}$ ← note order!

chain rule 5. $\frac{d}{dt}(\vec{F}(s(t))) = s'(t)\,\vec{F}'(s(t))$

Proofs: eg 4. $\frac{d}{dt}(F_iG_i) = \frac{dF_i}{dt}G_i + F_i\frac{dG_i}{dt}$

5. $\frac{d}{dt}(\vec{F}\times\vec{G})_i = \frac{d}{dt}(\epsilon_{ijk}F_jG_k) = \epsilon_{ijk}\left(\frac{dF_j}{dt}G_k + F_j\frac{dG_k}{dt}\right)$

$$= \epsilon_{ijk}\frac{dF_j}{dt}G_k + \epsilon_{ijk}F_j\frac{dG_k}{dt} = \left(\frac{d\vec{F}}{dt}\times\vec{G}\right)_i + \left(\vec{F}\times\frac{d\vec{G}}{dt}\right)_i$$

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eg Suppose $\vec{F}(t)$ has constant magnitude (varies only in direction)

$|\vec{F}(t)| = c$ ie. $\vec{F}(t)$ traces out a curve on the surface of a sphere of radius c.

Then either $\vec{F}'(t) = \vec{0}$ or $\vec{F}'(t) \perp \vec{F}(t)$

Proof. $|\vec{F}(t)| = \text{const} = c \Rightarrow |\vec{F}(t)|^2 = c^2$

$\Rightarrow \vec{F}(t)\cdot\vec{F}(t) = c^2$

$\Rightarrow \frac{d}{dt}|\vec{F}(t)|^2 = \frac{d}{dt}\vec{F}(t)\cdot\vec{F}(t) = 0$

$\Rightarrow \vec{F}(t)\cdot\frac{d\vec{F}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{F}(t) = 0 \qquad \Rightarrow 2\vec{F}\cdot\frac{d\vec{F}}{dt} = 0$

ie if $\frac{d\vec{F}}{dt} \neq \vec{0}$, then $\frac{d\vec{F}}{dt} \perp \vec{F}$ (tangent to curve is $\perp$ radius of sphere)

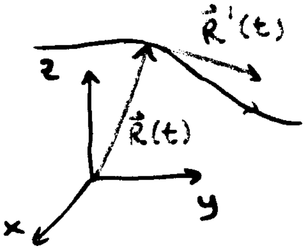
Space Curves

$$\vec{R} = \vec{R}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$$

$t_1 \le t \le t_2$ $\quad \vec{R}(t)$ continuous

$\vec{R}(t)$ is the position vector of a point in space

As t varies, $\vec{R}(t)$ traces out a space curve



space curve / arc / trajectory

a one-dimensional object: a single value t is needed to specify each point on the arc

t : parameter

$\vec{R}(t)$: parametrization of the curve

eg $\quad = \vec{R}_0 + t\vec{v}, \quad -\infty < t < \infty$ line

$\vec{R}_0$: fixed point on line

$\vec{v}$: vector parallel to line

$\vec{R}_1(t) = \vec{R}_0 + (\tan t)\,\vec{w}, \quad -\pi/2 < t < \pi/2, \quad \vec{w} = \frac{1}{3}\vec{v}$

alternative parametrization of line

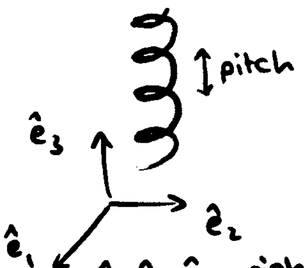
- $\vec{R}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$ circle radius 1, in x-y plane

$\vec{R}(t) = \vec{R}_0 + \rho\cos t\,\hat{\imath} + \rho\sin t\,\hat{\jmath}$ circle, center $\vec{R}_0$, radius ρ

(in plane $\perp \hat{k}$ ie $\parallel$ to x-y plane)

- a parametrization of $(x-x_0)^2 + (y-y_0)^2 = \rho^2, \quad z = z_0$

- $\vec{R}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$ helix — projection onto x-y plane is unit circle



$\vec{R}(t) = \vec{R}_0 + \rho\cos t\,\hat{e}_1 + \rho\sin t\,\hat{e}_2 + at\,\hat{e}_3$

helix, radius ρ, with axis through $\vec{R}_0$, parallel to $\hat{e}_3$

pitch $2\pi|a|$, $\quad$ right-handed / left-handed if $\begin{cases} a > 0 \\ a < 0 \end{cases}$

$\hat{e}_1, \hat{e}_2, \hat{e}_3$: right-handed system of unit vectors

Common interpretation: t: time, $\vec{R}(t)$: trajectory of particle position vector

In time interval $(t, t+\Delta t)$:

- Displacement $\boxed{\Delta\vec{R} = \vec{R}(t+\Delta t) - \vec{R}(t)} = \Delta x\,\hat{\imath} + \Delta y\,\hat{\jmath} + \Delta z\,\hat{k}$

$$\Rightarrow \frac{\Delta\vec{R}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k} \qquad \text{average velocity}$$

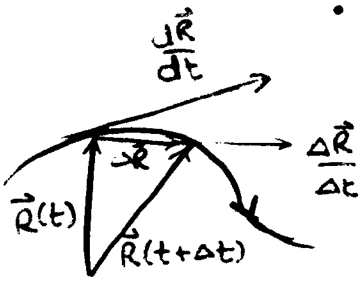
- (Instantaneous) Velocity $\boxed{\vec{v}(t) = \vec{R}'(t) = \frac{d\vec{R}}{dt}} = \lim_{\Delta t\to 0} \frac{\Delta\vec{R}}{\Delta t}$

$$\Rightarrow \vec{v}(t) = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

- Acceleration $\boxed{\vec{a}(t) = \frac{d^2\vec{R}}{dt^2} = \vec{R}''(t)} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}$

- Speed $\boxed{v(t) = |\vec{v}(t)|} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2}$

$\left(= \frac{ds}{dt}\right) \qquad = \left(\frac{dR_i}{dt}\frac{dR_i}{dt}\right)^{1/2}$



Note: $\vec{v}(t_0) = \frac{d\vec{R}}{dt}(t_0)$ is tangent to the curve $\vec{R} = \vec{R}(t)$ at $t = t_0$.

Unit tangent $\boxed{\vec{T}(t) = \frac{\vec{R}'(t)}{|\vec{R}'(t)|}} = \frac{\vec{v}(t)}{v(t)}$

$$\vec{T} = \frac{\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}}$$

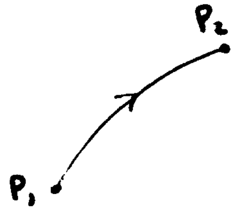
Defn: An arc is smooth if it has a parametrization $\vec{R} = \vec{R}(t)$, $t_1 \le t \le t_2$, satisfying: [possibly $t_1 = -\infty$, $t_2 = \infty$]

i) $\vec{R}'(t)$ exists and is continuous for $t_1 \le t \le t_2$

ii) No self-intersections: distinct t, $t_1 < t < t_2$, correspond to different points [ie $\vec{R}(\theta_1) = \vec{R}(\theta_2) \Rightarrow \theta_1 = \theta_2$, $t_1 < \theta_1, \theta_2 < t_2$]

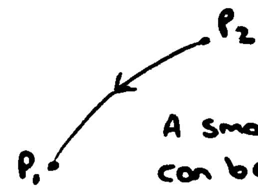
iii) $\vec{R}'(t) \neq \vec{0}$, $t_1 \le t \le t_2$
[ie there is a tangent at every point : no cusps]

eg



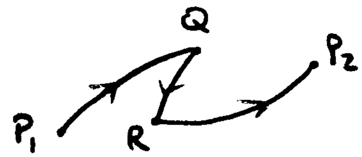
Smooth

<u>Oriented</u> (arrow shows direction of increasing t)



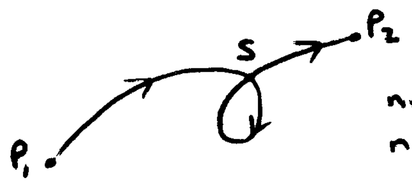
A smooth arc can be oriented in two ways

smooth, <u>closed</u>



not smooth (no tangent at Q, R)

<u>regular</u> (union of finite no. of smooth arcs, no self-intersections)



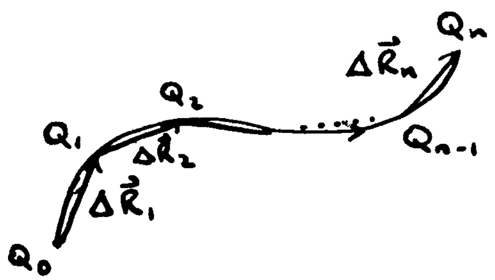
not smooth, not regular (crossing at S)

<u>Note</u>: The straight line $\vec{R}(t) = \vec{R}_0 + t^3 \vec{v}$, $-1 \le t \le 1$ violates condition iii), since $\vec{R}'(t) = \vec{0}$ at $t = 0$

<u>but</u> it is a smooth curve since it can be parametrized by $\vec{R}(t) = \vec{R}_0 + t\vec{v}$, $-1 \le t \le 1$

<u>Arc Length</u>

$C: \ \vec{R} = \vec{R}(t), \quad a \le t \le b$ smooth space curve



Partition $[a,b]$ into n subintervals

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$$

Q_j has position vector $\vec{R}(t_j)$, $0 \le j \le n$

Length of each segment

$$|\Delta \vec{R}_j| = |\vec{R}(t_j) - \vec{R}(t_{j-1})| = |\Delta x_j \,\hat{\imath} + \Delta y_j \,\hat{\jmath} + \Delta z_j \,\hat{k}|$$

$$= (\Delta x_j^2 + \Delta y_j^2 + \Delta z_j^2)^{1/2}$$

Mean value theorem: $\Delta x_j = x(t_j) - x(t_{j-1}) = \frac{dx}{dt}(\tau_j)\,\Delta t_j$

for some τ_j, $t_{j-1} \le \tau_j \le t_j$

$\Delta t_j = t_j - t_{j-1}$

and $\Delta y_j = \frac{dy}{dt}(\tau_j')\,\Delta t_j$, $\Delta z_j = \frac{dz}{dt}(\tau_j'')\Delta t_j$

Length of polygonal path

$$\sum_{j=1}^{n} |\Delta \vec{R}_j| = \sum_{j=1}^{n} \left[\left(\frac{dx}{dt}(\tau_j)\right)^2 + \left(\frac{dy}{dt}(\tau_j')\right)^2 + \left(\frac{dz}{dt}(\tau_j'')\right)^2 \right]^{1/2} \Delta t_j$$

Take limit as maximum length of subdivision $\to 0$: $\max\limits_{1\le j\le n} |\Delta t_j| \to 0$

Riemann integral:

$\Rightarrow$ Arc Length $\boxed{L = \int_a^b \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right]^{1/2} dt}$

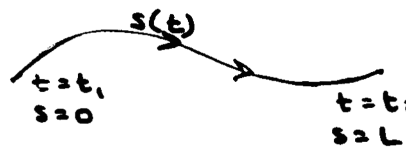
$$= \int_a^b \left|\frac{d\vec{R}}{dt}\right| dt$$

$$L = \int_C |d\vec{R}| = \int_C \left[dx^2 + dy^2 + dz^2\right]^{1/2}$$

$dx = \frac{dx}{dt}\,dt$ ...

arc length is a property of the curve, independent of parametrization

Arc length measured along curve



$$s = s(t) = \int_{t_1}^{t} \left|\frac{d\vec{R}}{dt}\right| dt \qquad t \ge t_1$$

Note: $\frac{ds}{dt} = \left|\frac{d\vec{R}}{dt}\right| = |\vec{v}(t)| = v(t)$: Speed = rate of change of arclength

" $ds = |d\vec{R}| = (dx^2 + dy^2 + dz^2)^{1/2}$ "

C a smooth curve $\Rightarrow \vec{R}'(t) \neq \vec{0}$ (for some parametrization)

$$\Rightarrow \frac{ds}{dt} = v(t) = |\vec{R}'(t)| > 0 \qquad (\neq 0)$$

ie the function $s = s(t)$ is monotonic, one-to-one

$\Rightarrow$ $s(t)$ can (in principle) be inverted to give $\underline{t = t(s)}$

$\Rightarrow$ Reparametrize – use s as a parameter, substitute $t(s)$

Arc length parametrization: $\vec{R}(s)$ (in principle; direct computation usually difficult)

Note: $$\frac{d\vec{R}}{dt} = \frac{d\vec{R}}{ds}\frac{ds}{dt} \Rightarrow \frac{d\vec{R}}{ds} = \frac{d\vec{R}}{dt} \Big/ \frac{ds}{dt} = \frac{1}{v}\frac{d\vec{R}}{dt}$$ tangent to curve

$$\left|\frac{d\vec{R}}{ds}\right| = \frac{|d\vec{R}/dt|}{ds/dt} = 1$$

$$\Rightarrow \boxed{\vec{T}(s) = \frac{d\vec{R}}{ds}} : \quad \text{unit tangent vector}$$

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eg Helix. $\vec{R}(t) = -\rho \sin t\,\hat{\imath} + \rho\cos t\,\hat{\jmath} + at\,\hat{k}$ $\quad 0 \le t \le 4\pi$
parametric form

(Note: $\hat{e}_1 = \hat{\jmath}$, $\hat{e}_2 = \hat{\imath}$, $\hat{e}_3 = -\hat{k}$ (right-handed) $\rho, a > 0$: left-handed helix)

[Equivalent non-parametric form: $x = \rho \sin z/a$, $y = \rho \cos z/a$]

$$\vec{v}(t) = \vec{R}'(t) = \rho\cos t\,\hat{\imath} - \rho \sin t\,\hat{\jmath} + a\,\hat{k}$$

$$v = |\vec{R}'(t)| = (\rho^2\cos^2 t + \rho^2 \sin^2 t + a^2)^{1/2} = (\rho^2 + a^2)^{1/2}$$

$$s = \int_0^t |\vec{R}'(t)|\,dt = \int_0^t (\rho^2+a^2)^{1/2}\,dt = (\rho^2+a^2)^{1/2}\,t$$

eg $\vec{R}(t) = 3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 4t\,\hat{k}$ $\qquad \rho = 3,\ a = 4$

$$\Rightarrow s = (3^2+4^2)^{1/2}t = 5t, \quad t = s/5$$

$$\Rightarrow \vec{R}(s) = 3\sin\tfrac{s}{5}\,\hat{\imath} + 3\cos\tfrac{s}{5}\,\hat{\jmath} + \tfrac{4}{5}s\,\hat{k}, \quad 0 \le s \le 20\pi$$

alternative parametrization

$$\vec{T}(s) = \vec{R}'(s) = \tfrac{3}{5}\cos\tfrac{s}{5}\,\hat{\imath} - \tfrac{3}{5}\sin\tfrac{s}{5}\,\hat{\jmath} + \tfrac{4}{5}\hat{k}, \quad |\vec{T}| = 1.$$

Acceleration and Curvature

Acceleration = rate of change of velocity :

Motivation for definitions, results : due to change in magnitude and/or direction

We know :

Change in magnitude only : motion along straight line

Tangential $|\vec{a}| = \left|\frac{d\vec{v}}{dt}\right| = \frac{d}{dt}|\vec{v}| = \frac{d^2 s}{dt^2}$

Change in direction only : motion along circle, radius ρ

Centripetal $|\vec{a}| = \frac{|\vec{v}|^2}{\rho} = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{v^2}{\rho}$

For general motion, acceleration is due to changes in both magnitude and direction.

For general curves: - what is ρ ?

Osculating circle best approximates curve at a given point P, tangent at P.

Radius ρ of osculating circle = radius of curvature

Smaller radius $\Leftrightarrow$ larger curvature

Characterizing curvature :

Unit tangent $\vec{T} = \frac{\vec{R}'}{|\vec{R}'|} = \frac{\vec{v}}{v} = \frac{d\vec{R}}{ds}$: fixed magnitude $|\vec{T}| = 1$

$\Rightarrow \frac{d\vec{T}}{ds}$ due only to changes in direction

Greater curvature $\Rightarrow$ greater change in $\vec{T}$

$\Rightarrow$ define

Curvature $\boxed{k = \left|\frac{d\vec{T}}{ds}\right|} = \frac{|d\vec{T}/dt|}{ds/dt} = \left|\frac{d^2\vec{R}}{ds^2}\right|$

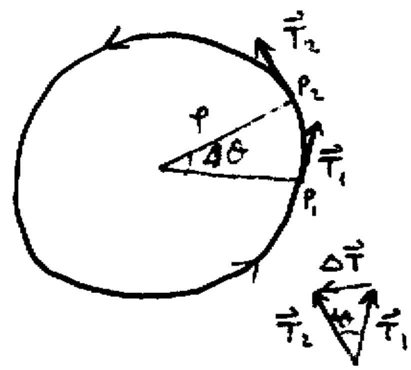
intrinsic properties of curves, independent of parametrization

Radius of curvature $\boxed{\rho = \frac{1}{k}} = 1 \Big/ \left|\frac{d\vec{T}}{ds}\right|$

(for $k \neq 0$)

rate of change of $\vec{T}$ w.r.t. arc length

[Check that the definitions agree with the usual radius for motion on a circle:



Arc Length from P_1 to P_2 : $\Delta s \approx \rho \, \Delta\theta$

Magnitude of change in unit tangent

$$|\Delta\vec{T}| = |\vec{T}_2 - \vec{T}_1| \approx \underbrace{|\vec{T}_1|}_{=1} \Delta\theta = \Delta\theta$$

$$\Rightarrow \left|\frac{\Delta\vec{T}}{\Delta s}\right| \approx \frac{\Delta\theta}{\Delta s} \approx \frac{\Delta\theta}{\rho\,\Delta\theta} = \frac{1}{\rho}$$

limit as $\Delta s \to 0$: $\quad k = \left|\dfrac{d\vec{T}}{ds}\right| = \dfrac{1}{\rho}$ — agrees with above definition.]

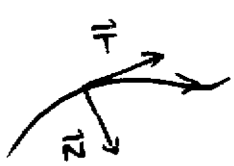
$\vec{T}$ has constant magnitude
$\Rightarrow$ either $\dfrac{d\vec{T}}{dt} = \vec{0}$ or $\dfrac{d\vec{T}}{dt} \perp \vec{T}$ $\quad \left[\begin{array}{l} \vec{T}\cdot\vec{T} = 1 \\ \Rightarrow \dfrac{d}{dt}\vec{T}\cdot\vec{T} = 2\vec{T}\cdot\dfrac{d\vec{T}}{dt} = 0 \end{array} \right.$

If $\dfrac{d\vec{T}}{dt} \neq 0$, define the unit vector

Principal normal $\quad \boxed{\vec{N} = \dfrac{d\vec{T}/dt}{|d\vec{T}/dt|}} = \dfrac{d\vec{T}/ds}{|d\vec{T}/ds|} = \dfrac{1}{k}\dfrac{d\vec{T}}{ds}$

chain rule

Note: $\quad \vec{T}\cdot\dfrac{d\vec{T}}{dt} = 0 \Rightarrow \vec{T}\cdot\vec{N} = 0 \quad$ ie $\quad \vec{N} \perp \vec{T}$



$\vec{N}$: a unit vector pointing to centre of osculating circle

Osculating plane : the plane containing $\vec{T}$ and $\vec{N}$

From the definition: $\quad \dfrac{d\vec{T}}{ds} = k\,\vec{N}$

ie $\vec{T}$ turns in the direction $\vec{N}$ at rate k per unit length of arc

Smooth arc $\vec{R}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ (Assume $\vec{R}(t) \in C^2$ ie $R_i(t)$ is twice continuously differentiable)

velocity $\Rightarrow$ $\vec{v}(t) = \vec{R}'(t) = \dfrac{dx}{dt}\hat{\imath} + \dfrac{dy}{dt}\hat{\jmath} + \dfrac{dz}{dt}\hat{k}$

speed $v(t) = |\vec{v}(t)| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2} = \dfrac{ds}{dt}$

acceleration $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \vec{R}''(t) = \dfrac{d^2x}{dt^2}\hat{\imath} + \dfrac{d^2y}{dt^2}\hat{\jmath} + \dfrac{d^2z}{dt^2}\hat{k}$

Derive an expression for $\vec{a}$ in terms of $\vec{T}, \vec{N}$:

Note $\vec{v}(t) = \dfrac{d\vec{R}}{dt} = \dfrac{d\vec{R}}{ds}\dfrac{ds}{dt}\left[= v\dfrac{d\vec{R}}{ds}\right] = \dfrac{ds}{dt}\vec{T}$

product rule

$$\Rightarrow \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right) = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt}$$

$\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds}\dfrac{ds}{dt}$

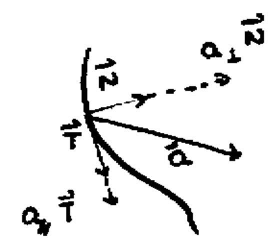
$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{ds}\frac{ds}{dt}$$

$\dfrac{d\vec{T}}{ds} = k\vec{N}$

$$= \frac{d^2s}{dt^2}\vec{T} + k\left(\frac{ds}{dt}\right)^2\vec{N}$$

ie $\boxed{\vec{a} = a_{/\!/}\vec{T} + a_{\perp}\vec{N}}$

$\vec{a}$ lies in the osculating plane



where (as expected)

$$a_{/\!/} = a_t = \frac{d^2s}{dt^2}, \qquad a_{\perp} = a_n = k\left(\frac{ds}{dt}\right)^2 = kv^2 = v^2/\rho$$

tangential acceleration — centripetal acceleration

Since $\vec{T} \perp \vec{N}$, we have (Pythagoras)

$$a^2 = |\vec{a}|^2 = a_{/\!/}^2 + a_{\perp}^2$$

- to find a, compute $d^2\vec{R}/dt^2$, calculate the magnitude
- to find $a_{/\!/}$, find $v = |d\vec{R}/dt| = \dfrac{ds}{dt}$, then differentiate wrt t
- then find $a_{\perp}$ from $a_{\perp}^2 = a^2 - a_{/\!/}^2$

eg $\vec{R}(t) = \underbrace{(\sin t - t\cos t)}_{x(t)}\hat{i} + \underbrace{(\cos t + t\sin t)}_{y(t)}\hat{j} + \underbrace{t^2}_{z(t)}\hat{k}$

$$\frac{d\vec{R}}{dt} = (t\sin t)\hat{i} + (t\cos t)\hat{j} + 2t\,\hat{k} = \vec{v}(t)$$

$$\frac{d^2\vec{R}}{dt^2} = (t\cos t + \sin t)\hat{i} + (-t\sin t + \cos t)\hat{j} + 2\hat{k} = \vec{a}(t)$$

Speed: $v(t) = \left|\frac{d\vec{R}}{dt}\right| = \frac{ds}{dt} = \left[t^2\sin^2 t + t^2\cos^2 t + 4t^2\right]^{1/2} = (5t^2)^{1/2} = \sqrt{5}\,t$

Tangential acceleration: $a_{\parallel} = \frac{d^2s}{dt^2} = \sqrt{5}$.

Acceleration: $a^2 = (t\cos t + \sin t)^2 + (-t\sin t + \cos t)^2 + 2^2$

$= t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t + t^2\sin^2 t - 2t\cos t\sin t + \cos^2 t + 4$

$= t^2 + 1 + 4 = t^2 + 5 \Rightarrow a = \sqrt{t^2+5}$

Centripetal acceleration: $a_\perp^2 = a^2 - a_{\parallel}^2 = t^2 + 5 - 5 = t^2 \Rightarrow a_\perp = t$

Curvature: $a_\perp = kv^2 \Rightarrow k = a_\perp / v^2 = t/5t^2 = 1/5t$

Radius of curvature: $\rho = 1/k = 5t$

Tangent vector: $\vec{T}(t) = \frac{\vec{R}'(t)}{v(t)} = \frac{1}{\sqrt{5}}\sin t\,\hat{i} + \frac{1}{\sqrt{5}}\cos t\,\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$

Principal normal: $\vec{N}(t) = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{\frac{1}{\sqrt{5}}\cos t\,\hat{i} - \frac{1}{\sqrt{5}}\sin t\,\hat{j}}{1/\sqrt{5}} = \cos t\,\hat{i} - \sin t\,\hat{j}$

Arc length: $s(t) = \int_0^t \sqrt{5}\,t'\,dt' = \frac{\sqrt{5}}{2}t^2 \Rightarrow t = \frac{\sqrt{2s}}{5^{1/4}}$

Another formula for curvature:

$$\vec{R}' = v\vec{T}, \qquad \vec{R}'' = \frac{d^2s}{dt^2}\vec{T} + kv^2\vec{N}$$

since $\vec{T} = \frac{\vec{R}'}{|\vec{R}'|} = \frac{\vec{R}'}{v}$

$\vec{a} = \underset{\frac{d^2s}{dt^2}}{a_{\parallel}}\vec{T} + \underset{kv^2}{a_\perp}\vec{N}$

$$\Rightarrow \vec{R}' \times \vec{R}'' = v\vec{T} \times \left[\frac{d^2s}{dt^2}\vec{T} + kv^2\vec{N}\right] = kv^3\,\vec{T}\times\vec{N}$$

Define Binormal $\boxed{\vec{B} = \vec{T}\times\vec{N}}$ – since $|\vec{T}| = |\vec{N}| = 1$, $\vec{T} \perp \vec{N}$, we have $|\vec{B}| = |\vec{T}\times\vec{N}| = 1$

$$\Rightarrow \vec{R}'\times\vec{R}'' = kv^3\,\vec{T}\times\vec{N} = kv^3\vec{B}$$

$$\Rightarrow |\vec{R}'\times\vec{R}''| = kv^3|\vec{B}| = k\,|\vec{R}'|^3$$

$$\Rightarrow \boxed{k = \frac{|\vec{R}'\times\vec{R}''|}{|\vec{R}'|^3}} = \frac{|\vec{v}\times\vec{a}|}{v^3} \quad \text{and} \quad a_\perp = kv^2 = \frac{|\vec{R}'\times\vec{R}''|}{|\vec{R}'|}$$

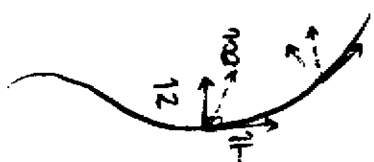
Frenet Formulas (Frenet - Serret) - some differential geometry...

Unit tangent $\vec{T} = \dfrac{d\vec{R}}{ds}$, curvature $k = \left|\dfrac{d\vec{T}}{ds}\right| = \left|\dfrac{d^2\vec{R}}{ds^2}\right|$

Principal normal $\vec{N} = \dfrac{1}{k}\dfrac{d\vec{T}}{ds}$ $\Rightarrow |\vec{T}| = 1, \quad |\vec{N}| = 1, \quad \vec{T}\cdot\vec{N} = 0$

Binormal $\vec{B} = \vec{T}\times\vec{N}$ $\Rightarrow |\vec{B}| = 1 \quad \vec{B}\cdot\vec{T} = \vec{B}\cdot\vec{N} = 0$

$\vec{T}, \vec{N}, \vec{B}$ form a right-handed system of orthonormal vectors : Frenet frame



As we move along the curve, the triad of unit vectors moves and rotates

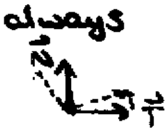
Motivation for Frenet formulas: (not a proof)

$$\frac{d\vec{T}}{ds} = \underbrace{k\vec{N}}_{①}$$

$\vec{T}$ turns towards $\vec{N}$ at rate ① k

$\Rightarrow \vec{N}$ turns towards $-\vec{T}$ at rate ② k

since $\vec{T}\perp\vec{N}$ always



Also, $\vec{N}$ could rotate about $\vec{T}$: tilt of osculating plane instantaneously containing curve (twist)

$\longrightarrow$ τ : torsion : measures rate at which curve twists

ie $\dfrac{d\vec{N}}{ds}$ could have a component $\perp \vec{T}$

(and $\perp \vec{N}$: $\vec{N}\cdot\vec{N} = 1 \Rightarrow \vec{N}\cdot\dfrac{d\vec{N}}{ds} = 0$) ie ③ $/\!/ \vec{B}$

Thus

$$\frac{d\vec{N}}{ds} = \underbrace{-k\vec{T}}_{②} + \underbrace{\tau\vec{B}}_{③} \text{ - twist}$$

$\vec{N}$ turns towards $\vec{B}$ at rate τ ③ $\Rightarrow \vec{B}$ turns towards $\vec{N}$ at rate $-\tau$ ④

$\vec{B}$ does not turn towards $\vec{T}$, since $\vec{T}$ turns only towards $\vec{N}$, by definition

Thus

$$\frac{d\vec{B}}{ds} = \underbrace{-\tau\vec{N}}_{④}$$

Derivation of Frenet formulas:

$$\boxed{\begin{aligned} \frac{d\vec{T}}{ds} &= \kappa \vec{N} \\ \frac{d\vec{N}}{ds} &= -\kappa \vec{T} + \tau \vec{B} \\ \frac{d\vec{B}}{ds} &= -\tau \vec{N} \end{aligned}}$$

1. $\underline{\dfrac{d\vec{T}}{ds} = \kappa \vec{N}}$, and $\underset{\text{curvature}}{\kappa} = \left|\dfrac{d\vec{T}}{ds}\right| \geq 0$ (definition).

2. • $\vec{B} = \vec{T} \times \vec{N} \Rightarrow \vec{B}\cdot\vec{B} = |\vec{B}|^2 = 1$

$$\Rightarrow \vec{B}\cdot\frac{d\vec{B}}{ds} = \frac{1}{2}\frac{d}{ds}(\vec{B}\cdot\vec{B}) = 0 \quad \Rightarrow \frac{d\vec{B}}{ds} \perp \vec{B}$$

• $\dfrac{d\vec{B}}{ds} = \dfrac{d}{ds}(\vec{T}\times\vec{N}) = \dfrac{d\vec{T}}{ds}\times\vec{N} + \vec{T}\times\dfrac{d\vec{N}}{ds} = \kappa \overbrace{\cancel{\vec{N}\times\vec{N}}}^{0} + \vec{T}\times\dfrac{d\vec{N}}{ds}$

$= \vec{T}\times\dfrac{d\vec{N}}{ds}$, so $\dfrac{d\vec{B}}{ds} \perp \vec{T}$.

Since $\dfrac{d\vec{B}}{ds} \perp \vec{B}$, and $\perp \vec{T}$, we must have $\dfrac{d\vec{B}}{ds} \parallel \vec{N}$

Thus we can write $\underline{\dfrac{d\vec{B}}{ds} = -\tau \vec{N}}$ ← this formula defines the torsion τ.

and $\boxed{\tau = -\vec{N}\cdot\dfrac{d\vec{B}}{ds}}$ scalar ("amount of twist")

3. $\vec{T}, \vec{N}, \vec{B}$ form a right-handed system

$\Rightarrow \vec{N} = \vec{B}\times\vec{T}$

$$\Rightarrow \frac{d\vec{N}}{ds} = \frac{d\vec{B}}{ds}\times\vec{T} + \vec{B}\times\frac{d\vec{T}}{ds} = (-\tau)\underbrace{\vec{N}\times\vec{T}}_{-\vec{B}} + \kappa\,\underbrace{\vec{B}\times\vec{N}}_{-\vec{T}}$$

$$\Rightarrow \underline{\frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B}}$$

∎

Notes:

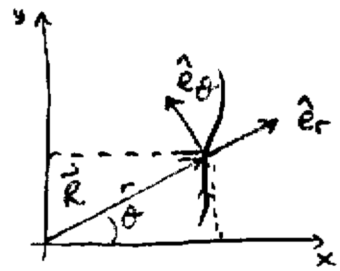
- For planar motion, $\vec{T}, \vec{N}$ lie in the plane of the curve, $\vec{B}$ is a constant unit vector $\perp$ plane, torsion $\tau = 0$ (no twist)

$\kappa = \kappa(s)$, $\tau = \tau(s)$ intrinsic equations of curve

- Two curves with identical $\kappa(s)$, $\tau(s)$ are congruent (can coincide after a rigid motion) $\Rightarrow \kappa(s), \tau(s)$ describe curve

- Formulas for torsion: $\tau = \dfrac{[\vec{R}', \vec{R}'', \vec{R}''']}{|\vec{R}'\times\vec{R}''|^2} = \dfrac{1}{\kappa^2}\left[\dfrac{d\vec{R}}{ds}, \dfrac{d^2\vec{R}}{ds^2}, \dfrac{d^3\vec{R}}{ds^3}\right]$

Planar Motion in Polar Coordinates



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \begin{aligned} r &\geq 0 \\ -\pi < \theta &\leq \pi \end{aligned}$$

$$\Rightarrow \quad r = (x^2+y^2)^{1/2}$$

$$\tan\theta = y/x \Rightarrow \theta = \arctan\frac{y}{x} = \arcsin\frac{y}{(x^2+y^2)^{1/2}}$$

avoid quadrant ambiguities

Radial and angular unit vectors $\hat{e}_r, \hat{e}_\theta$ (or $\hat{u}_r, \hat{u}_\theta$)

vary from point to point

$$\left.\begin{aligned} \hat{e}_r &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\ \hat{e}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \end{aligned}\right\}$$

not constant!

Note $\hat{e}_r\cdot\hat{e}_r = \hat{e}_\theta\cdot\hat{e}_\theta = 1$
$\hat{e}_r\cdot\hat{e}_\theta = 0$

$$\frac{\partial\hat{e}_r}{\partial r} = 0, \quad \frac{\partial\hat{e}_\theta}{\partial r} = 0, \quad \frac{\partial\hat{e}_r}{\partial\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \hat{e}_\theta, \quad \frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r$$

Chain rule

$$\Rightarrow \quad \frac{d\hat{e}_r}{dt} = \frac{\partial\hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial\hat{e}_r}{\partial\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\hat{e}_\theta, \qquad \frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\hat{e}_r$$

Motion in polar coordinates:

$\vec{R}(t) = r\,\hat{e}_r$ position vector of a particle at (r,θ)

velocity

$$\vec{v}(t) = \frac{d\vec{R}}{dt} = \frac{dr}{dt}\hat{e}_r + r\underbrace{\frac{d\hat{e}_r}{dt}}_{\frac{d\theta}{dt}\hat{e}_\theta}$$

$$\Rightarrow \quad \vec{v}(t) = \underbrace{\frac{dr}{dt}\hat{e}_r}_{\text{radial}} + \underbrace{r\frac{d\theta}{dt}\hat{e}_\theta}_{\text{transverse}}$$

acceleration

$$\vec{a}(t) = \frac{d^2\vec{R}}{dt^2} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\underbrace{\frac{d\hat{e}_r}{dt}}_{\frac{d\theta}{dt}\hat{e}_\theta} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta + r\frac{d\theta}{dt}\underbrace{\frac{d\hat{e}_\theta}{dt}}_{-\frac{d\theta}{dt}\hat{e}_r}$$

$$= \left[\frac{d^2r}{dt^2} - r\frac{d\theta}{dt}\right]\hat{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{e}_\theta$$

Velocity $\vec{v} = \frac{d\vec{R}}{dt} = \underbrace{\frac{dr}{dt}\hat{e}_r}_{\text{radial}} + \underbrace{r\frac{d\theta}{dt}\hat{e}_\theta}_{\text{tangential}}$

Acceleration $\vec{a} = \frac{d^2\vec{R}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{e}_\theta$

$\frac{d^2r}{dt^2}\hat{e}_r$: acceleration for pure radial motion

$r\frac{d^2\theta}{dt^2}\hat{e}_\theta$: pure angular acceleration

$-r\left(\frac{d\theta}{dt}\right)^2\hat{e}_r$: centripetal acceleration (if r = constant)

$2\frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta$: related to Coriolis acceleration [due to change in direction of $\hat{e}_r$, & transverse component changes with r even if $\frac{d\theta}{dt}$ = const.]

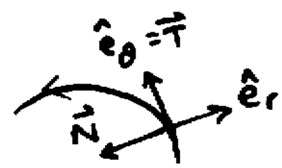
Special cases:

1. θ = constant, $\vec{R} = r\hat{e}_r$ ← constant

$$\Rightarrow \vec{a} = \frac{d^2r}{dt^2}\hat{e}_r = \frac{d^2s}{dt^2}\vec{T} \qquad (\text{since } s = r)$$

- one-dimensional motion (straight line)

2. r = constant - circular motion



$$\vec{a} = -r\left(\frac{d\theta}{dt}\right)^2\hat{e}_r + r\frac{d^2\theta}{dt^2}\hat{e}_\theta$$

$$= r\left(\frac{d\theta}{dt}\right)^2\vec{N} + r\frac{d^2\theta}{dt^2}\vec{T}$$

$$\left[\text{and } s = r\theta \Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt},\ \frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2}\right]$$

$$\Rightarrow \vec{a} = k\left(\frac{ds}{dt}\right)^2\vec{N} + \frac{d^2s}{dt^2}\vec{T}$$

$$\left[\text{curvature } k = \frac{1}{r}\right]$$

Newton's 2nd Law: $\vec{F} = m\vec{a}$

$$\vec{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta$$ write force i.t.o. radial, angular components

$$\Rightarrow F_r = m\frac{d^2r}{dt^2} - mr\left(\frac{d\theta}{dt}\right)^2 = m\ddot{r} - mr\dot{\theta}^2$$ time derivative

[convention: $\dot{r} \equiv \frac{dr}{dt}$

and

$$F_\theta = mr\frac{d^2\theta}{dt^2} + 2m\frac{dr}{dt}\frac{d\theta}{dt}$$

$$\Rightarrow \underbrace{rF_\theta}_{\text{torque}} = mr^2\frac{d^2\theta}{dt^2} + 2mr\frac{dr}{dt}\frac{d\theta}{dt} = \frac{d}{dt}\Big[\underbrace{mr^2\frac{d\theta}{dt}}_{\text{angular momentum}}\Big]$$

Applied torque = time rate of change of angular momentum

[In general: Newton's 2nd Law $\vec{F} = m\frac{d\vec{v}}{dt}$

Torque $\vec{T} = \vec{R}\times\vec{F} = \vec{R}\times m\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{R}\times m\vec{v})$ $\left(\text{since } \underbrace{\frac{d\vec{R}}{dt}}_{\vec{v}}\times m\vec{v} = \vec{0}\right)$

$= \frac{d\vec{L}}{dt}$, where $\vec{L} = \vec{R}\times\vec{p} = \vec{R}\times(m\vec{v})$ is the angular momentum

In this case $\vec{T} = r\hat{e}_r\times(F_r\hat{e}_r + F_\theta\hat{e}_\theta) = rF_\theta\,\hat{e}_r\times\hat{e}_\theta = rF_\theta\,\hat{k}$

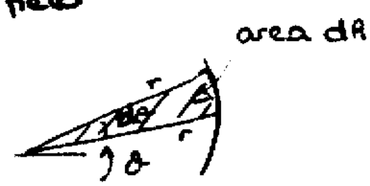
and $\vec{L} = \vec{R}\times m\vec{v} = mr\hat{e}_r\times\left(\frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta\right) = mr^2\frac{d\theta}{dt}\,\hat{k}$]

If $F_\theta = 0$ ie $\vec{F} = F_r\hat{e}_r$: radial (central) force

then $\frac{d}{dt}\left[mr^2\frac{d\theta}{dt}\right] = 0 \Rightarrow mr^2\frac{d\theta}{dt} = \text{const} = C$

angular momentum is constant in a central force field

Geometric interpretation:



$dA = \frac{1}{2}r^2 d\theta$

infinitesimal area as radius vector rotates through angle $d\theta$

$$\Rightarrow A(t) = \int_{t_1}^{t_2}\frac{1}{2}r^2\frac{d\theta}{dt}\,dt \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}$$

For a particle moving under a central force, equal areas are swept out in equal times: Kepler's 2nd Law of planetary motion