# 3. Scalar and Vector Fields

General function $f : \mathbb{R}^m \to \mathbb{R}^n$ (n functions of m variables)

Impotant cases:

Vector function of a single variable $\vec{f} : \mathbb{R} \to \mathbb{R}^n$

Scalar field $f : \mathbb{R}^n \to \mathbb{R}$

Vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$

(usually $n = 2$ or $3$)

we consider $n = 3$ – everything specializes to $n = 2$.

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## Scalar fields

$f(x, y, z)$ : scalar field

real - (scalar) - valued function of 3 variables

- write $f(x,y,z) = f(\vec{x})$
  where $\vec{r} = \vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the position vector of the point $(x, y, z)$.

- associate a scalar to each point $(x,y,z) \in D \subset \mathbb{R}^3$
  domain of $f$

Examples: $f$ could represent mass density; temperature; pressure; gravitational or electrostatic potential

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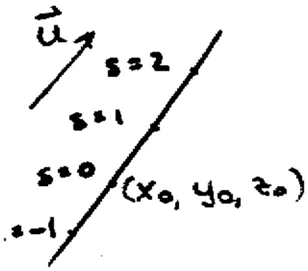
$f(x,y,z) = c$ = constant : defines a surface :

isotimic surface (level surface)

eg if $f(x,y,z)$ is
- temperature : isothermal surface
- pressure : isobaric surface
- electrostatic potential : equipotential surface

## Directional derivative:

Consider the behaviour of the scalar field $f(x,y,z)$ in the neighbourhood of a point $(x_0, y_0, z_0)$:




Choose a vector $\vec{u}$; consider a line segment through $(x_0,y_0,z_0)$ parallel to $\vec{u}$

s: distance along line segment measured in direction of $\vec{u}$

$s=0$ at $(x_0, y_0, z_0)$ ie at $\vec{x} = \vec{x}_0$.

$\vec{x}(s) = \vec{x}_0 + s\,\vec{u}$ : position vectors of points on the line

$s \mapsto f(\vec{x}(s)) = f(x(s), y(s), z(s))$ : f evaluated along line segment, as function of distance s. (function of one variable)

Rate of change of f w.r.t. s?

Defn:

The directional derivative of f at $(x_0, y_0, z_0)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(\vec{x}_0) \equiv \left.\frac{df}{ds}\right|_{s=0} = \left.\frac{d}{ds} f(\vec{x}_0 + s\vec{u})\right|_{s=0} = \lim_{s\to 0} \frac{f(\vec{x}_0 + s\vec{u}) - f(\vec{x}_0)}{s}$$

(if the derivative exists)

– rate of change of f in the prescribed direction.

Convention: choose $\vec{u}$ to be a unit vector

Position vector of points along line segment $\vec{x}(s) = \vec{x}_0 + s\vec{u}$

$$\Rightarrow \vec{x}(s) = \vec{R}(s) = x(s)\,\hat{i} + y(s)\,\hat{j} + z(s)\,\hat{k}$$

$\Rightarrow$ unit tangent

$$\vec{u} = \frac{d\vec{x}}{ds} = \frac{d\vec{R}}{ds} = \frac{dx}{ds}\,\hat{i} + \frac{dy}{ds}\,\hat{j} + \frac{dz}{ds}\,\hat{k}$$

Chain rule: (if partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ exist, are continuous)

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$\Rightarrow D_{\vec{u}} f = \frac{df}{ds} = \underbrace{\left[\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right]}_{\text{grad } f = \nabla f} \cdot \underbrace{\left[\frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + \frac{dz}{ds}\hat{k}\right]}_{\vec{u}}$$

<u>Gradient</u>:

The <u>gradient</u> of $f$ is defined to be the vector

$$\boxed{\text{grad } f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = \nabla f}$$

using "del" notation

Note: if $f(x,y,z)$ is a scalar field, grad $f$ is a vector field.

$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ (or $\vec{\nabla}$)

Then

$$\boxed{D_{\vec{u}} f = \frac{df}{ds} = \vec{u}\cdot\text{grad } f} = \vec{u}\cdot\nabla f$$

directional derivative

<u>Properties</u>:

1. $D_{\vec{u}} f = \frac{df}{ds} = \vec{u}\cdot\nabla f = \underbrace{|\vec{u}|}_{=1 \text{ (unit vector)}}|\nabla f|\cos\theta = |\nabla f|\cos\theta$

   $\theta$ angle between $\nabla f$ and $\vec{u}$

   $\Rightarrow$ the component of grad $f$ in any given direction gives the directional derivative $\frac{df}{ds}$ in that direction

   eg $D_{\hat{\imath}} f = \hat{\imath}\cdot\nabla f = \frac{\partial f}{\partial x}$ : the partial derivative $\frac{\partial f}{\partial x}$ is the directional derivative parallel to the x-axis, in the direction $\hat{\imath}$ (vary x, keep y and z constant). Similarly $D_{\hat{\jmath}} f = \frac{\partial f}{\partial y}$, $D_{\hat{k}} f = \frac{\partial f}{\partial z}$

2. $|\vec{u}| = 1 \Rightarrow$ largest value of $\frac{df}{ds}$ occurs when $\cos\theta = 1$ ie when $\vec{u}$ and $\nabla f$ are in the same direction :

   maximum value of $D_{\vec{u}} f$ occurs in the direction of $\nabla f$

   $\Rightarrow \nabla f$ points in the direction of <u>maximum</u> <u>rate of increase</u> of the function $f$. ($-\nabla f$ : maximum <u>decrease</u> of $f$)

3. If $\vec{u} \parallel \text{grad } f \Rightarrow \cos\theta = 1$, we have $\frac{df}{ds} = D_{\vec{u}} f = |\text{grad } f|$,

$\Rightarrow$ the magnitude of $\nabla f$ equals the maximum rate of increase of $f$ per unit distance

4. Let $S$ be the level surface $f(x,y,z) = c = \text{const}$, and let $(x_0, y_0, z_0)$ lie in $S$ ie $f(x_0, y_0, z_0) = c$.

Consider a path $\vec{x}(s)$ contained in $S$, with $\vec{x}(0) = \vec{x}_0$, and with tangent vector $\vec{v} = \vec{x}'(0)$ at $s=0$.

Then $\quad \vec{v} \cdot \nabla f(\vec{x}_0) = 0 \qquad \leftarrow$ holds for every arc through $\vec{x}_0$

$\Rightarrow$ the gradient vector $\nabla f$ is normal to the isotimic surface (if $\nabla f(\vec{x}_0) \neq \vec{0}$)

[since if $\vec{x}(s)$, $s_1 \le s \le s_2$ lies in the level surface $S$, then $f(\vec{x}(s)) = c$

$$\Rightarrow 0 = \frac{d}{ds} f(\vec{x}(s))\Big|_{s=0} = \nabla f(\vec{x}(0)) \cdot \vec{x}'(0) = \nabla f(\vec{x}_0) \cdot \vec{v}$$

$\Rightarrow \vec{v} \perp \nabla f(\vec{x}_0)$.]

[The converse holds: if $\nabla f(\vec{x}_0) \neq \vec{0}$, then there is an isotimic surface $f(x,y,z) = c$ through $(x_0, y_0, z_0)$, and grad $f$ is perpendicular to this surface.]

Tangent plane:

If $\nabla f(\vec{x}_0) \neq \vec{0}$, the tangent plane to $S$ is given by

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = \nabla f(x_0, y_0, z_0) \cdot \left[(x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}\right] = 0$$

5. The gradient vanishes, grad $f = \vec{0}$, at any local maximum (or minimum) of a continuously differentiable scalar field.

Examples:

1. $f(x,y,z) = x + 2y - 3z$ $\quad [f = \text{const} \Leftrightarrow \vec{R}\cdot\vec{n} = \text{const.}]$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k} \quad \leftarrow \text{constant } \nabla f$$

Isotimic surfaces $f$ = const. are planes with normal $\vec{n} = \nabla f = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

2. $f(x,y,z) = x^2+y^2+z^2$, $\quad \nabla f = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k} = 2\vec{R}$

Spheres, centre at origin : level surfaces

radius vector points away from O

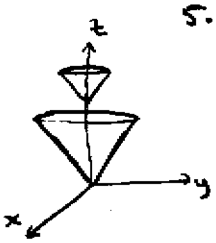
3. $f(x,y,z) = x^2 + y^2$, $\quad \nabla f = 2x\hat{\imath} + 2y\hat{\jmath}$ $\quad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

level surfaces: cylinders (axis of symmetry $z$-axis) : $x^2 + y^2 = \rho^2 > 0$

$$\Rightarrow \nabla f = 2\rho(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = 2\rho\,\hat{e}_\rho$$

4. $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2$, $\quad \nabla f = \frac{x}{2}\hat{\imath} + \frac{2}{9}y\hat{\jmath} + 2z\hat{k}$

isotimic surfaces: family of ellipsoids




5. $f(x,y,z) = \sqrt{x^2+y^2} - z = \rho - z$, $\quad \nabla f = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath} - \hat{k}$

isotimic surfaces are conical. $\quad |\nabla f| = \left(\frac{x^2+y^2}{x^2+y^2} + 1\right)^{1/2} = \sqrt{2}$

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Different isotimic surfaces of the same scalar field do not intersect.

Note: $g(x,y)$ - a function of two variables
$\Rightarrow$ the graph of the function $z = g(x,y)$ is an isotimic surface (with $c=0$) of the function $f(x,y,z) = z - g(x,y)$.

Summary:
grad $f = \nabla f$ gives
- direction of maximum increase of $f$
- direction orthogonal to level surfaces of $f$

# Vector Fields and Flow Lines

Vector field $\vec{F}$ in $\mathbb{R}^n$: a map $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$
vector-valued function

$n=2$: vector field in the plane; $n=3$: in space

$\Rightarrow$ a vector field in space $\vec{F}$ associates a vector $\vec{F}(x,y,z)$ to each point $(x,y,z)$ in some region $D \subset \mathbb{R}^3$.
(visualize a vector (arrow) extending from each point)

A vector field $\vec{F}$ has three component scalar fields $F_1, F_2, F_3$:

$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$$

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eg $\vec{F}$: velocity field
$\vec{F}(\vec{x})$ is the velocity in a fluid at position $\vec{x}$
(in general, velocity $\vec{v}(\vec{x},t)$ depends on $\vec{x}$ and time $t$.
Steady flow: velocity at each point does not change with time)

eg The gradient of any scalar field $f$ is a vector field

$$\vec{G} = \nabla f$$

(important in many applications)

$\Rightarrow$ gradient vector field.

eg Gravitational field

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\vec{r}$$

$\leftarrow$ attractive force of a particle at the origin, mass $M$, on another particle, mass $m$, position vector $\vec{r}$

$\vec{F}$ is a gradient field:

$$\vec{F} = -\nabla V, \qquad V = -\frac{GMm}{r} \quad \leftarrow \text{gravitational potential}$$

Check: $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x = \frac{x}{r}$

$\frac{\partial V}{\partial x} = -GMm\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -GMm\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} = GMm\frac{x}{r^3}$; similarly $\frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$

$\Rightarrow -\nabla V = -\frac{GMm}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{GMm}{r^3}\vec{r} = -\frac{GMm}{r^2}\hat{r} = \vec{F}$

eg <u>Coulomb's Law</u>

Force of charge $Q$ (at origin) on charge $q$, position vector $\vec{r}$ :

$$\vec{F} = \frac{kQq}{r^2}\hat{r} = \frac{kQq}{r^3}\vec{r} = q\vec{E}$$

| $k$ : constant, depends on units

$\vec{E} = \frac{kQ}{r^3}\vec{r}$ : force per unit charge exerted at position $\vec{r}$ by the charge $Q$ :
<u>electric field</u> due to charge $Q$

$\vec{E} = -\nabla V$, $\quad V = \frac{kQ}{r}$ : electrostatic potential (voltage)

(The electric field and force field are orthogonal to equipotential surfaces, which are spheres in this case.)

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Not every vector field is a gradient vector field

eg $\vec{F} = y\hat{\imath} - x\hat{\jmath}$ ← not a gradient field

⌈since suppose $\vec{F} = \nabla g$ for some scalar field $g$

ie $\vec{F} = \frac{\partial g}{\partial x}\hat{\imath} + \frac{\partial g}{\partial y}\hat{\jmath} \Rightarrow \frac{\partial g}{\partial x} = y, \ \frac{\partial g}{\partial y} = -x$

but then $\frac{\partial^2 g}{\partial y\,\partial x} = 1, \ \frac{\partial^2 g}{\partial x\,\partial y} = -1$ which violates the equality of mixed partial derivatives

→ no such $g$ exists ie $\vec{F}$ is not a gradient field. ⌋

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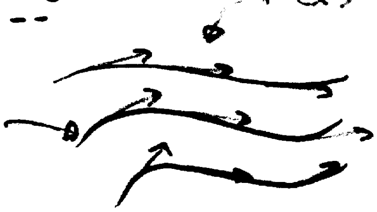
<u>Flow Lines</u> $\vec{F}(\vec{x})$ : vector field

flow lines



Flow line: streamline
integral curve
characteristic curve
if $\vec{F}$ is a force field → <u>line of force</u>

A <u>flow line</u> of a vector field $\vec{F}$ is any curve so that at each point on the curve, $\vec{F}$ is tangent to the curve.

Interpretation: A flow line (streamline) is the path traced by a particle whose velocity at each point has the same direction as that of the vector field $\vec{F}$.

eg if $\vec{F} = \vec{v}$ is the steady (time-independent) velocity field of a fluid, then a flow line (streamline) is the path traced out by a small particle suspended in the fluid.

Notes: 1. $g$ a scalar field, $g(\vec{x}) \neq 0$ for each $\vec{x}$
$\Rightarrow$ flow lines of $g(\vec{x})\vec{F}(\vec{x})$ coincide with those of $\vec{F}(x)$

2. Since $\vec{F}(\vec{x})$ uniquely determines the direction at each point $\vec{x}$, we cannot have two different flow directions at any point $\Rightarrow$ flow lines do not cross

3. $\vec{F}(\vec{x}_0) = \vec{0}$ for some $\vec{x}_0$ $\Rightarrow$ no direction defined at $\vec{x}_0$
$\Rightarrow$ no flow line through $\vec{x}_0$

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$\vec{x} = \vec{R}$ : position vector of a point on the flow line

$s$ : arclength $\Rightarrow$ unit tangent $\vec{T} = \dfrac{d\vec{R}}{ds} = \dfrac{dx}{ds}\hat{\imath} + \dfrac{dy}{ds}\hat{\jmath} + \dfrac{dz}{ds}\hat{k}$

Tangent is parallel to flow line $\Rightarrow$ $\vec{T} = \beta\vec{F}$ (for some constant $\beta$)

$$\Rightarrow \frac{dx}{ds} = \beta F_1 , \quad \frac{dy}{ds} = \beta F_2, \quad \frac{dz}{ds} = \beta F_3$$

$F_1, F_2, F_3$ all nonzero

$$\Rightarrow \frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} = \beta\, ds$$

$$\Rightarrow \int \frac{dx}{F_1} = \int \frac{dy}{F_2} = \int \frac{dz}{F_3}$$ : equations for flow lines

eg $f(x,y,z) = x^2 + y^2$

- find flow lines of the gradient vector field

(recall: level sets are cylinders about the $z$-axis)

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 0\,\hat{k}$$

$$\Rightarrow \underbrace{\frac{dx}{ds} = 2\beta x \; , \; \frac{dy}{ds}} = 2\beta y \; , \quad \frac{dz}{ds} = 0 \Rightarrow z = \text{constant}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

integrate

$$\Rightarrow \ln|x| = \ln|y| + \text{const} = \ln a|y|$$

$$\Rightarrow |x| = A|y| \qquad (\text{some constant } A > 0)$$

$$\Rightarrow x = cy \qquad (\text{any constant } c) \qquad \text{and } z = \text{const.}$$

ie flow lines are straight half-lines, parallel to $x$-$y$ plane, extending outwards from $z$-axis.

Note: flow lines are perpendicular to level sets of $f$.

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Recall: the tangent to flow lines is parallel to $\vec{F}$

$$\frac{d\vec{R}}{ds} = \frac{d\vec{x}}{ds} = \beta\vec{F} \qquad (\beta \neq 0)$$

← flow line $\vec{x}(s)$ parametrized by arc length

Introduce the new parameter $t = \beta s$

$$\Rightarrow \frac{d\vec{x}}{ds} = \frac{d\vec{x}}{dt}\frac{dt}{ds} = \beta\frac{d\vec{x}}{dt} = \beta\vec{F}$$

Thus: the flow lines (streamlines, integral curves) are paths such that (in some parametrization)

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}(t))$$

← system of differential equations

ie
$$\left.\begin{aligned} x'(t) = \frac{dx}{dt} &= F_1(x(t), y(t), z(t)) \\ y'(t) = \frac{dy}{dt} &= F_2(x(t), y(t), z(t)) \\ z'(t) = \frac{dz}{dt} &= F_3(x(t), y(t), z(t)) \end{aligned}\right\}$$

Determine a solution uniquely by specifying initial condition $\vec{x}(0) = \vec{x}_0$

Solution: function $\vec{x}(t)$ s.t. tangent $\vec{x}'(t)$ is everywhere given by vector field

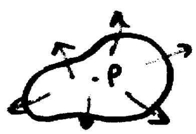
# Divergence

Gradient: measures rates of change of a <u>scalar</u> field $f$

Divergence, Curl $\}$ measure changes of a <u>vector</u> field $\vec{F}$

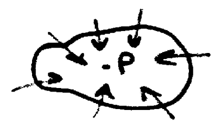
Divergence of a vector field: a scalar field that (roughly) measures how a field diverges:

"the amount by which the flow lines out of a unit volume exceed the flow lines into the unit volume".



div $\vec{F} > 0$ at $P$

expansion: $P$ is a "source"



div $\vec{F} < 0$ at $P$

compression / contraction: $P$ is a "sink"

If $\vec{F}$ is a velocity field in space, div $\vec{F}$ measures the rate of expansion per unit volume under the flow

(in the plane, div $\vec{F}$ gives the rate of expansion of area).

$$\boxed{\text{Divergence: "flux per unit volume"}}$$

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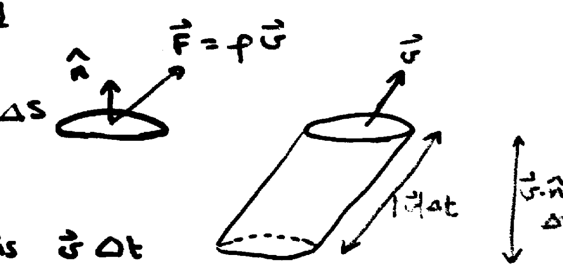
Consider a fluid: velocity field $\vec{v}$ ($\vec{v}(\vec{x})$: velocity of particle at $\vec{x} = (x,y,z)$)

mass density $\rho$ (mass per unit volume)

Define the vector field $\vec{F} = \rho \vec{v}$ $\qquad \vec{F}(\vec{x}) = \rho(\vec{x})\,\vec{v}(\vec{x})$

<u>mass flux density</u>

Consider a small planar surface, area $\Delta S$, unit normal $\hat{n}$:




In time interval $\Delta t$, displacement is $\vec{v}\,\Delta t$

$\Rightarrow$ volume of fluid flowing through $\Delta S$ is volume of cylinder: base $\Delta S$, height $(\vec{v}\cdot\hat{n})\Delta t$

$\Rightarrow$ volume $= \underbrace{(\vec{v}\cdot\hat{n})}_{\text{normal component of velocity}} \Delta S\, \Delta t$

Amount of fluid in cylinder in time interval $\Delta t$

= mass of fluid flowing through area $\Delta S$ in time $\Delta t$ $= (\vec{v}\cdot\hat{n})\rho\,\Delta S\,\Delta t$

$\Rightarrow$ mass flow rate (per unit time) $= \rho(\vec{v}\cdot\hat{n})\Delta S = \vec{F}\cdot\hat{n}\,\Delta S$

<u>Flux</u> of vector field $\vec{F}$ through small area $\Delta S$ with outward unit normal $\hat{n}$

(in general, flux of $\vec{F}$ through surface $S$ is $\iint_S \vec{F}\cdot\hat{n}\,dS$ - see later)

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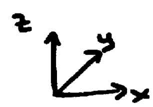
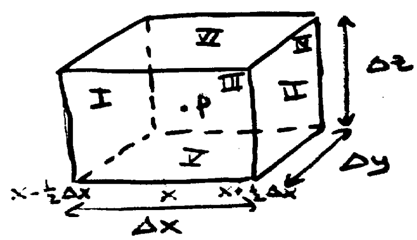
Now compute divergence as flux per unit volume, in <u>Cartesian coordinates</u>:

Consider a rectangular parallelepiped,
sides $\Delta x$, $\Delta y$, $\Delta z$,
volume $\Delta V = \Delta x\,\Delta y\,\Delta z$
centre $P: (x, y, z)$




mathematical statement of "flux per unit volume" $\rightarrow$ $$\boxed{\operatorname{div}\vec{F} = \lim_{V\to 0}\frac{1}{V}\iint_{\partial V}\vec{F}\cdot\hat{n}\,dS}$$ where $V$ is a small volume enclosed by the surface $\partial V$ (we will explain this later)

Compute the total flux of $\vec{F}$ through the surface of the box, in the out<u>ward</u> direction on each face; divide by the volume, take limit as dimensions $\Delta x, \Delta y, \Delta z \to 0$.

Vector field $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

Consider faces I, II:

evaluate $\vec{F}$ at centre of each face: good approximation for small $\Delta x, \Delta y, \Delta z$

I : outward normal $-\hat{\imath}$, flux $\underbrace{-\vec{F}(x-\tfrac{\Delta x}{2},y,z)\cdot\hat{\imath}}_{F_1}\ \underbrace{\Delta y\,\Delta z}_{\Delta S}$

II : outward normal $\hat{\imath}$, flux $+\vec{F}(x+\tfrac{\Delta x}{2},y,z)\cdot\hat{\imath}\ \Delta y\,\Delta z$

Total flux through I and II:

$$\approx \frac{F_1(x+\frac{\Delta x}{2},y,z) - F_1(x-\frac{\Delta x}{2},y,z)}{\Delta x}\,\Delta x\,\Delta y\,\Delta z \underset{\Delta x\to 0}{\approx} \left.\frac{\partial F_1}{\partial x}\right|_{(x,y,z)}\Delta x\,\Delta y\,\Delta z$$

Similarly, net flux through faces III, IV $\approx \dfrac{\partial F_2}{\partial y}\,\Delta x\,\Delta y\,\Delta z$

net flux through faces V, VI $\approx \dfrac{\partial F_3}{\partial z}\,\Delta x\,\Delta y\,\Delta z$

$$\Rightarrow \text{net outward flux} \approx \underbrace{\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)}_{\text{div } \vec{F}} \underbrace{\Delta x\,\Delta y\,\Delta z}_{\Delta V}$$

$\Rightarrow$ flux per unit volume (with $\Delta V \to 0$) is $\dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

<u>Defn</u>: (in Cartesian coordinates)

The <u>divergence</u> of a vector field $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is the scalar field $\text{div}\,\vec{F}$ defined by (Cartesian)

$$\boxed{\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}} = \nabla \cdot \vec{F}$$

"del" notation : later

If $\text{div}\,\vec{F} = 0$ everywhere, the vector field $\vec{F}$ is "divergence-free" or <u>solenoidal</u>.

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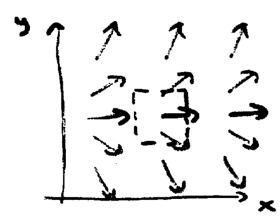
eg $\vec{x} = \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ : $\text{div}\,\vec{R} = 3$

eg




$F_3 \equiv 0$, no $z$-variation

"flow lines diverging"

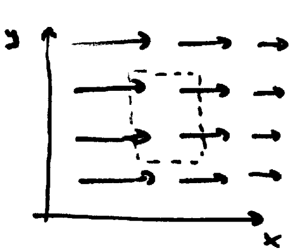
Expect $\text{div}\,\vec{F} > 0$ : flow through $x$-faces cancels, but flow <u>out</u> of $y$-faces.

Check: $F_1 \approx \text{const} \Rightarrow \dfrac{\partial F_1}{\partial x} \approx 0$

$F_2$ increasing with $y \Rightarrow \dfrac{\partial F_2}{\partial y} > 0$

$F_3 \equiv 0 \Rightarrow$ <u>$\text{div}\,\vec{F} > 0$</u>

eg




Expect $\text{div}\,\vec{F} < 0$ (even though flow lines are parallel) : more fluid enters region from left than leaves it to right.

Check: $F_2 \equiv 0$, $F_3 \equiv 0$, $\dfrac{\partial F_1}{\partial x} < 0 \Rightarrow \text{div}\,\vec{F} < 0$

## Conservation Laws, Equation of Continuity

$\vec{v}(\vec{x},t)$ : velocity of fluid at position $\vec{x}$, time $t$. (vector field)

$\rho(\vec{x},t)$ : mass density (scalar field)

$\Rightarrow$ mass flux density $\vec{F}(\vec{x},t) = \rho(\vec{x},t)\,\vec{v}(\vec{x},t)$

- Total flux of $\vec{F}$ through surface of small box with dimensions $\Delta x, \Delta y, \Delta z$, volume $\Delta V$, centred at $\vec{x} = (x,y,z)$, is (mass flowing out)
  flux $= (\operatorname{div} \vec{F})\,\Delta V = \operatorname{div}(\rho\vec{v})\,\Delta V$
  $=$ rate of decrease of mass inside box (per unit time)

- Total mass of fluid in volume $\Delta V$ at time $t$ is $\rho(\vec{x},t)\,\Delta V$

**Conservation of Mass:** rate of decrease of mass inside volume = flux out of volume through surface

mass is not created or destroyed

ie $$-\frac{\partial}{\partial t}\left[\rho(\vec{x},t)\,\Delta V\right] = (\operatorname{div}\vec{F})\,\Delta V = \operatorname{div}(\rho\vec{v})\,\Delta V$$

$$\Rightarrow\ -\frac{\partial \rho}{\partial t} = \operatorname{div}(\rho\vec{v})$$

$$\Rightarrow\ \boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0}$$

Equation of continuity

(in fluid mechanics: mathematical expression of mass conservation)

Many common fluids, such as water, are "incompressible":

to a good approximation, $\rho \simeq$ constant

Then the continuity equation becomes $\operatorname{div}\vec{v} = 0$

incompressibility: divergence-free velocity fields.

---

In general, if $\mu(\vec{x},t)$ represents the <u>density</u> of a physical quantity, and $\vec{F}(\vec{x},t)$ gives the <u>flux</u> density of that quantity, then conservation of the quantity (no sources or sinks) is represented by the equation $$\frac{\partial \mu}{\partial t} + \operatorname{div}\vec{F} = 0$$

eg charge conservation:
$$\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{J} = 0 \quad \begin{cases} \rho : \text{charge} \\ \vec{J} : \text{current density} \end{cases}$$

## Curl

Divergence of a vector field: related to expansion (contraction of small volumes

Curl of a vector field: related to rotation

$$\boxed{\text{Curl : "circulation per unit area"}}$$

In general, the circulation of a vector field $\vec{F}$ around a curve C is $\Gamma_C = \oint_C \vec{F}\cdot d\vec{\ell} = \oint_C \vec{F}\cdot\vec{T}\,ds$

Curl $\vec{F}$ is a vector field; if A is a small area whose boundary is the closed curve $C = \partial A$, and A has unit normal vector $\hat{n}$, the component of curl $\vec{F}$ in the direction of $\hat{n}$ is

mathematical statement of "circulation per unit area"

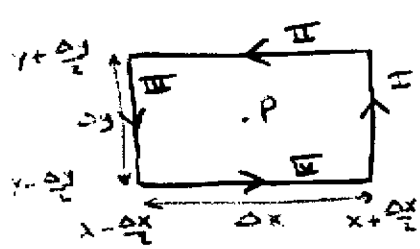
$$\boxed{(\text{curl } \vec{F})\cdot\hat{n} = \lim_{A\to 0} \frac{1}{A}\oint_C \vec{F}\cdot d\vec{\ell}}$$

- see later

---

As before, let $\vec{F} = \rho\vec{v}$ : $\vec{v}$ : velocity field, $\rho$ : density, $\vec{F}$ : mass flux density

To find the z-component of curl $\vec{F}$, ie $(\text{curl } \vec{F})\cdot\hat{k}$, we consider circulation in the x-y plane, with normal $\hat{k}$.
(measures mass flow rate around P)
Consider a small rectangle, C, centered at P: (x, y, z), parallel to the x-y plane, with sides $\Delta x$, $\Delta y$




(right-hand rule: follow C in counter-clockwise direction, keeping the enclosed area to the left.)

Circulation: lengths of edges weighted by the tangential (counterclockwise) components of $\vec{F}$ along edges (evaluated at centres of edges - good approximation since $\Delta x, \Delta y$ small)

Circulation "swirl"
$$\approx \underbrace{F_2(x+\tfrac{\Delta x}{2}, y, z)}_{\mathrm{I}}\Delta y + \underbrace{\left[-F_1(x, y+\tfrac{\Delta y}{2}, z)\right]}_{\mathrm{II}}\Delta x + \underbrace{\left[-F_2(x-\tfrac{\Delta x}{2}, y, z)\right]}_{\mathrm{III}}\Delta y + \underbrace{F_1(x, y-\tfrac{\Delta y}{2}, z)}_{\mathrm{IV}}\Delta x$$

$$= \frac{F_2(x+\frac{\Delta x}{2}, y, z) - F_2(x-\frac{\Delta x}{2}, y, z)}{\Delta x}\,\Delta x\,\Delta y$$
$$- \frac{F_1(x, y+\frac{\Delta y}{2}, z) - F_1(x, y-\frac{\Delta y}{2}, z)}{\Delta y}\,\Delta x\,\Delta y$$

$\Delta x, \Delta y \to 0$
$$\approx \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underbrace{\Delta x\,\Delta y}_{\text{area of rectangle}}$$

$\Rightarrow$ circulation (swirl) per unit area normal to $z$ direction is $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ : gives $z$-component of curl

$$(\text{curl}\,\vec{F})\cdot\hat{k} = (\text{curl}\,\vec{F})_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Similarly
$$(\text{curl}\,\vec{F})\cdot\hat{\imath} = (\text{curl}\,\vec{F})_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$$
← use circulation per unit area in $y$-$z$ plane

$$(\text{curl}\,\vec{F})\cdot\hat{\jmath} = (\text{curl}\,\vec{F})_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}$$

<u>Def</u>$^n$: (in Cartesian coordinates)

The <u>curl</u> of a vector field $\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$

is the vector field curl $\vec{F}$ defined by (Cartesian)

$$\boxed{\text{curl}\,\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}}$$

$= \nabla \times \vec{F}$ ← later: del notation

Write as <u>symbolic determinant</u>:

in Cartesian coordinates

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

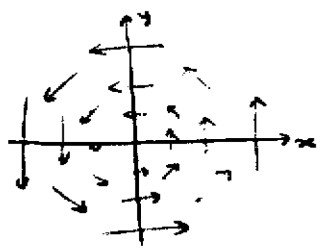
eg Fluid flowing with circular motion, with uniform angular velocity $\omega$ about the $z$-axis

(expect circular motion $\Rightarrow$ nonzero curl).

$\Rightarrow$ angular velocity $\vec{\omega} = \omega\hat{k}$

$\Rightarrow$ velocity

$$\vec{v} = \vec{\omega}\times\vec{R} = \omega\hat{k}\times(x\hat{i}+y\hat{j}+z\hat{k})$$
$$= \omega x(\underbrace{\hat{k}\times\hat{i}}_{\hat{j}}) + \omega y(\underbrace{\hat{k}\times\hat{j}}_{-\hat{i}}) + \omega z(\underbrace{\hat{k}\times\hat{k}}_{\vec{0}})$$
$$= -\omega y\,\hat{i} + \omega x\,\hat{j}$$



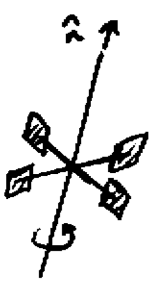
$$\Rightarrow \operatorname{curl}\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix} = \hat{i}\cdot 0 + \hat{j}\cdot 0 + \hat{k}(\omega+\omega)$$
$$= 2\omega\hat{k}$$

$$\Rightarrow \operatorname{curl}\vec{v} = 2\vec{\omega}$$

$\leftarrow$ this vector result holds for any $\vec{\omega}$: we are free to choose the $z$-axis to be the axis of rotation

$\Rightarrow$ the curl of a velocity field of constant rotation is twice the angular velocity.

Angular velocity / rotational interpretation of curl:

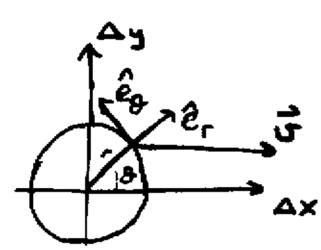



Consider a paddle wheel immersed in a fluid, velocity field $\vec{v}$: the paddle wheel rotates in the presence of a nonzero average angular velocity around its axis.

Assume axis in $z$-direction $\Rightarrow$ rotation due to velocities in $x$-$y$ plane

Polar coordinate system about axis:

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$




Centre of circle at $(x,y,z)$

$\Delta x = r\cos\theta$

$\Delta y = r\sin\theta$

Velocity field

$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

Counterclockwise component of velocity

$$\vec{v}\cdot\hat{e}_\theta = -v_1\sin\theta + v_2\cos\theta \quad \text{, angular velocity } \frac{\vec{v}\cdot\hat{e}_\theta}{r}$$

Consider variations of $\vec{v}$ near $(x,y,z)$, to first order in $\Delta x, \Delta y$ (Taylor expansion)

$$v_1(x+\Delta x, y+\Delta y, z) \approx v_1(x,y,z) + \frac{\partial v_1}{\partial x}(x,y,z)\,\Delta x + \frac{\partial v_1}{\partial y}(x,y,z)\,\Delta y + (\text{quadratic in } \Delta x, \Delta y)$$

$\Delta x = r\cos\theta,\ \Delta y = r\sin\theta$

$$\Rightarrow\ v_1 + \frac{\partial v_1}{\partial x}\, r\cos\theta + \frac{\partial v_1}{\partial y}\, r\sin\theta + O(r^2)$$

similarly

$$v_2(x+\Delta x, y+\Delta y, z) = v_2 + \frac{\partial v_2}{\partial x}\, r\cos\theta + \frac{\partial v_2}{\partial y}\, r\sin\theta + O(r^2)$$

evaluated at $(x,y,z)$

$$\Rightarrow\ \vec{v}\cdot\hat{e}_\theta = -v_1\sin\theta - r\frac{\partial v_1}{\partial x}\cos\theta\sin\theta - r\frac{\partial v_1}{\partial y}\sin^2\theta + v_2\cos\theta + r\frac{\partial v_2}{\partial x}\cos^2\theta + r\frac{\partial v_2}{\partial y}\sin\theta\cos\theta + O(r^2)$$

depends on $\theta$

$\Rightarrow$ average counterclockwise component around a small circle, radius $r$

is $$\langle \vec{v}\cdot\hat{e}_\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi} \vec{v}\cdot\hat{e}_\theta\, d\theta = \frac{r}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) + O(r^2)$$

(since average value of $\sin\theta$, $\cos\theta$, $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ over a complete period is zero: $\int_0^{2\pi}\sin\theta\, d\theta = 0$, ...
and average value of $\sin^2\theta, \cos^2\theta$ is $\frac{1}{2}$: $\int_0^{2\pi}\sin^2\theta\, d\theta = \int_0^{2\pi}\cos^2\theta\, d\theta = \pi$)

$\Rightarrow$ angular velocity of fluid about $z$-axis, at $(x,y,z)$, is

$$\frac{1}{r}\langle\vec{v}\cdot\hat{e}_\theta\rangle = \frac{1}{2}\underbrace{\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)} = \frac{1}{2}\underbrace{(\mathrm{curl}\,\vec{v})\cdot\hat{k}}$$

take limit as $r\to 0$

$z$ component of curl $\vec{v}$

(similarly for rotation about $x, y$ axes)

$\Rightarrow$ the component of <u>curl $\vec{v}$</u> in the direction $\hat{n}$ is twice the average angular velocity of rotation of the fluid about an axis parallel to $\hat{n}$.

[In view of this interpretation: old notation rot $\vec{v}$ for the curl]

$\hat{n}\cdot \text{curl}\,\vec{v}$ is a maximum when $\hat{n}$ points in the direction of $\text{curl}\,\vec{v}$; then $\hat{n}\cdot\text{curl}\,\vec{v} = |\text{curl}\,\vec{v}|$

Rotational interpretation of curl:

- the direction of $\text{curl}\,\vec{v}$ is the axis about which the fluid is rotating most rapidly;
- the magnitude $|\text{curl}\,\vec{v}|$ is twice the angular velocity about that axis.

Note: in fluid dynamics: $\vec{v}$ — velocity field
$\text{curl}\,\vec{v}$ — vorticity field (usually denoted $\vec{\omega}$)

If $\text{curl}\,\vec{F} = \vec{0}$ everywhere, the vector field $\vec{F}$ is "curl-free" or irrotational.

eg $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$: $\text{curl}\,\vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$
$= \vec{0}$

Scalar curl:

Let $\vec{F} = F_1(x,y)\,\hat{i} + F_2(x,y)\,\hat{j}$ be a vector field in the plane ($\mathbb{R}^2$)

$\Rightarrow$ $\vec{F}$ can be regarded as a vector field in space ($\mathbb{R}^3$) with $\hat{k}$ component $F_3 = 0$, and no z-dependence, $\frac{\partial}{\partial z} = 0$

$\Rightarrow \text{curl}\,\vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$ ← always in $\hat{k}$ direction.

$\Rightarrow$ $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is the scalar curl of the planar vector field $\vec{F}$.

function of x, y →

# Functions, Operators and Del notation

**Function** $f$: maps a number to a number

(scalar-valued) • a rule which associates with every number $x$ (in its domain of definition) a single real number $f(x)$

$f$: the rule

$f(2)$: the value of the function at $x=2$

eg if $f(x) = 2x^2 - 3$: $f: x \mapsto f(x) = 2x^2 - 3$

$f: 2 \mapsto f(2) = 5$

"maps to"

More general functions

eg Vector field $\vec{F}$: maps a position vector $\vec{x}$ (a point $(x,y,z)$) to a single vector $\vec{F}(\vec{x})$

etc.

We can generalize the idea of rules and maps:

**Operator** $T$: maps a function to a function

• a rule which associates with each function $f$ (in its domain of definition) some function $T(f)$

eg $\frac{d}{dx}$ : derivative operator : with each differentiable function $f$ it associates its derivative $\frac{df}{dx}$

$D$

$$\frac{d}{dx} : f \mapsto \frac{df}{dx} \qquad \left(\text{or } D(f) = \frac{df}{dx}\right)$$

[ $\frac{d}{dx}$ takes a function, gives a new function ]

$\frac{d}{dx}$ is a linear operator:

$$\frac{d}{dx}(f_1 + f_2) = \frac{df_1}{dx} + \frac{df_2}{dx}$$

$$\frac{d}{dx}(cf) = c\frac{df}{dx} \qquad (\text{constant scalar } c)$$

A linear operator $\mathcal{L}$ satisfies

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$$

(for functions $f_1, f_2$, constants $c_1, c_2$)

Linear differential operators

eg $\mathcal{L} = \frac{d^2}{dx^2} + 3\frac{d}{dx} + 2$

maps a function $f$ to $\mathcal{L}f = \frac{d^2f}{dx^2} + 3\frac{df}{dx} + 2f = f'' + 3f' + 2f$

$\mathcal{L} : f \mapsto \mathcal{L}f$

(the property of linearity is fundamental to the solution theory of linear differential equations, of the form $\mathcal{L}y = 0$

eg for $\mathcal{L}$ as above, $\mathcal{L}y = 0 \Leftrightarrow y'' + 3y' + 2y = 0$ ⊛

if $y_1(x)$, $y_2(x)$ are solutions, then by linearity, so are all functions $c_1 y_1(x) + c_2 y_2(x)$ for constants $c_1, c_2$

eg for equation ⊛, $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$ are solutions,

$\Rightarrow$ general solution is $y(x) = c_1 e^{-x} + c_2 e^{-2x}$ )

---

**Vector differential operators:**

Gradient : maps a scalar field to a vector field

$f \mapsto \text{grad } f$

Divergence : maps a vector field to a scalar field

$\vec{F} \mapsto \text{div } \vec{F}$

Curl : maps a vector field to a vector field

$\vec{F} \mapsto \text{curl } \vec{F}$

Laplacian : maps a $\left\{\begin{matrix}\text{scalar}\\ \text{vector}\end{matrix}\right\}$ field to a $\left\{\begin{matrix}\text{scalar}\\ \text{vector}\end{matrix}\right\}$ field

$f \mapsto \Delta f = \nabla^2 f, \quad \vec{F} \mapsto \Delta \vec{F} = \nabla^2 \vec{F}$

**Laplacian** It is frequently useful to introduce a single operator, the composite of the div and grad operators :

Laplacian (frequently written $\Delta$ or $\nabla^2$)

usually in mathematics — justified below

Laplacian of a scalar field $f$ : ie of $f(x,y,z)$ (Cartesian coordinates)

$$\Delta f = \text{Laplacian}(f) = \text{div}(\text{grad } f) = \text{div}\left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$\nabla^2 f \quad = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We can take $\Delta$ or $\nabla^2$ to be a convenient abbreviation for the (scalar) differential operator $\boxed{\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$

In this sense, the Laplacian can also operate on vector fields: $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

$$\Delta \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2} = (\nabla^2 F_1)\hat{\imath} + (\nabla^2 F_2)\hat{\jmath} + (\nabla^2 F_3)\hat{k}$$

$$\nabla^2 \vec{F}$$

The Laplacian plays an important role in many physical laws; we shall study properties and applications later.

---

**Del operator** – a convenient symbolic notation

Let $\boxed{\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$ "del", "nabla"

Then (in Cartesian coordinates) applying $\nabla$ as if it were a vector

write $\vec{\nabla} f$ (vector!) $\longrightarrow$ $\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \text{grad } f$

$$\nabla \cdot \vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{div } \vec{F}$$

$$\vec{\nabla}\times\vec{F} \quad \nabla\times\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(F_1\hat{\imath}+F_2\hat{\jmath}+F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \text{curl } \vec{F}$$

$$\nabla^2 = \nabla\cdot\nabla = \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = \text{Laplacian of } f$$

Summary:

- It is convenient to work with $\nabla$ as if it were a vector (but take note of order, as the del operator acts on functions appearing to its right : $\underbrace{\nabla\cdot\vec{G}}_{\text{scalar field}} \neq \underbrace{\vec{G}\cdot\nabla}_{\text{scalar differential operator}}$ )

- With $\nabla = \hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$ ,
  the expressions $\nabla f$, $\nabla\cdot\vec{F}$, $\nabla\times\vec{F}$, $\nabla^2 f$ give the formulas for grad $f$, div $\vec{F}$, curl $\vec{F}$, Laplacian of $f$ respectively in Cartesian coordinates

- We shall see that these operators take very different forms in other coordinate systems ; however, we will often write $\nabla f$ for grad $f$, $\nabla\cdot\vec{F}$ for div $\vec{F}$, $\nabla\times\vec{F}$ for curl $\vec{F}$, $\nabla^2 f$ for the Laplacian of $f$
  (slight abuse of notation)

## Tensor Notation

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ : vector operator

Conventions:

- Coordinates $(x, y, z) \longrightarrow (x_1, x_2, x_3)$
- Unit basis vectors $(\hat{\imath}, \hat{\jmath}, \hat{k}) \longrightarrow (\hat{e}_1, \hat{e}_2, \hat{e}_3)$
- Summation convention $\vec{F} = F_i \hat{e}_i$
  where $(\vec{F})_i = F_i$, $(f\vec{G})_i = fG_i$ etc.
- Write $\partial_i$ for $\frac{\partial}{\partial x_i}$ (abbreviation) $i = 1, 2, 3$

(in some contexts it is common to write $f_{,i}$ for $\partial_i f = \frac{\partial f}{\partial x_i}$)

$$\Rightarrow \nabla = \hat{e}_i \frac{\partial}{\partial x_i} = \hat{e}_i \partial_i$$

(recall: $\vec{F}\cdot\vec{G} = F_i G_i$, $(\vec{F}\times\vec{G})_i = \epsilon_{ijk} F_j G_k$ )

Then

gradient: $(\text{grad } f)_i = (\nabla f)_i = \partial_i f = f_{,i}$

divergence: $\text{div } \vec{F} = \nabla\cdot\vec{F} = \partial_i F_i = F_{i,i}$

curl: $(\text{curl } \vec{F})_i = (\nabla\times\vec{F})_i = \epsilon_{ijk}\partial_j F_k = \epsilon_{ijk} F_{k,j}$

Laplacian: $\Delta f = \nabla^2 f = \partial_i \partial_i f = f_{,ii}$

$\partial_i^2 f$ (apply summation convention to squared terms)

---

## Vector Operator Identities

Let $\vec{F}, \vec{G}$ be differentiable vector fields

$f, g$ be differentiable scalar fields

$\vec{A}$ : constant vector, $a, b$ constant scalars

Linearity of vector differential operators:

$$\left.\begin{aligned}
&\text{i)} && \nabla(af + bg) = a\nabla f + b\nabla g \\
&\text{ii)} && \nabla\cdot(a\vec{F} + b\vec{G}) = a\nabla\cdot\vec{F} + b\nabla\cdot\vec{G} \\
&\text{iii)} && \nabla\times(a\vec{F} + b\vec{G}) = a\nabla\times\vec{F} + b\nabla\times\vec{G}
\end{aligned}\right\}\ \text{immediate from definitions}$$

Product rules (involving scalar fields) (read as)

iv) $\nabla(fg) = f\,\nabla g + g\nabla f$ | $\text{grad}(fg) = f\,\text{grad}\,g + g\,\text{grad}f$

v) $\nabla\cdot(f\vec{a}) = f\,\nabla\cdot\vec{a} + \vec{a}\cdot\nabla f$ | $\text{div}(f\vec{a}) = f\,\text{div}\,\vec{a} + \vec{a}\cdot\text{grad}\,f$

vi) $\nabla\times(f\vec{a}) = f\,\nabla\times\vec{a} + \nabla f\times\vec{a}$ | $\text{curl}(f\vec{a}) = f\,\text{curl}\,\vec{a} + \text{grad}f\times\vec{a}$

These follow immediately from the ordinary product rule for functions of one variable, as seen by writing the equations out componentwise.

Using tensor notation

eg v) $\nabla\cdot(f\vec{a}) = \partial_i(f\vec{a})_i = \partial_i(fa_i) = f\,\partial_i a_i + a_i\,\partial_i f$

product rule for $\partial_i = \partial/\partial x_i$ $\quad = f\,\nabla\cdot\vec{a} + \vec{a}\cdot\nabla f$ ∎

eg vi) $[\nabla\times(f\vec{a})]_i = \epsilon_{ijk}\,\partial_j(f\vec{a})_k = \epsilon_{ijk}\,\partial_j(fa_k)$

$= \epsilon_{ijk}(\partial_j f)a_k + \epsilon_{ijk}\,f\,\partial_j a_k = [(\nabla f)\times\vec{a}]_i + f[\nabla\times\vec{a}]_i$ ∎

---

Note: $\vec{a}\cdot\nabla$ is a scalar differential operator:

$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z} = a_i\,\partial_i$$

$\vec{a}\cdot\nabla f$ is unambiguous (it can be interpreted in two ways, which are equivalent => parentheses unnecessary)

since $\vec{a}\cdot(\nabla f) = \vec{a}\cdot\left(\hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) = a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z}$

$= a_i\partial_i f$ $\qquad = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)f = (\vec{a}\cdot\nabla)f$

We interpret $(\vec{a}\cdot\nabla)\vec{F}$ as the (scalar) operator $\vec{a}\cdot\nabla$ acting on each component of the vector field $\vec{F}$:

$$[(\vec{a}\cdot\nabla)\vec{F}]_i = a_j\,\partial_j F_i\,.$$

---

Chain rule ($q$: a differentiable function of one variable)

$f, g$: scalar fields

vii) $\nabla q(f) = \dfrac{dq}{df}\nabla f = q'(f)\nabla f$

viii) $\Rightarrow \nabla(f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$ (quotient rule)

## Identities involving second derivatives

$f, g$ twice differentiable ($C^2$) scalar fields, $\vec{F}$: $C^2$ vector field

ix) $\nabla \times (\nabla f) = \vec{0}$ — curl (grad $f$) $= \vec{0}$

x) $\nabla \cdot (\nabla \times \vec{F}) = 0$ — div (curl $\vec{F}$) $= 0$

xi) $\nabla \cdot (\nabla f \times \nabla g) = 0$

xii) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ — curl(curl $\vec{F}$) = grad(div $\vec{F}$) − laplacian $\vec{F}$

These are based on the equality of continuous mixed second-order partial derivatives eg $\partial_i \partial_j f = \partial_j \partial_i f$ if $f \in C^2$

(first three identities vanish due to symmetry under interchange of differentiation operators, and antisymmetry in cross product $\times$ )

eg ix) $\boxed{\nabla \times \nabla f = \vec{0}}$ The curl of any gradient is zero: Gradients are curl-free (irrotational)

(symbolically: "$\nabla \times \nabla = 0$" compare $\vec{A} \times \vec{A} = \vec{0}$ for any vector $\vec{A}$ )

Proof:

$$[\nabla \times (\nabla f)]_i = \epsilon_{ijk}\, \partial_j (\nabla f)_k = \epsilon_{ijk}\, \partial_j \partial_k f \underset{\uparrow}{=} -\epsilon_{ikj}\, \partial_j \partial_k f$$

property of permutation tensor $\epsilon_{ijk}$

equality of mixed partials

$$= -\epsilon_{ikj}\, \partial_k \partial_j f = -\epsilon_{ijk} \partial_j \partial_k f = -[\nabla \times (\nabla f)]_i$$

exchange dummy variables $j, k$

$$\Rightarrow [\nabla \times (\nabla f)]_i = 0 \quad , \quad i = 1,2,3$$

∎

x) $\boxed{\nabla \cdot \nabla \times \vec{F} = 0}$ The divergence of any curl is zero: Curls are divergence-free (solenoidal)

( "$[\nabla, \nabla, \vec{F}] = 0$" scalar triple product: cf. $[\vec{A}, \vec{A}, \vec{B}] = 0$ with repeated term )

Proof:

$$\nabla \cdot \nabla \times \vec{F} = \partial_i (\nabla \times \vec{F})_i = \partial_i\, \epsilon_{ijk}\, \partial_j F_k = \epsilon_{ijk}\, \partial_i \partial_j F_k$$

$$= -\epsilon_{jik} \partial_i \partial_j F_k = -\epsilon_{jik}\, \partial_j \partial_i F_k \underset{i \leftrightarrow j}{=} -\epsilon_{ijk} \partial_i \partial_j F_k$$

$$= -\nabla \cdot \nabla \times \vec{F}$$

$$\Rightarrow \nabla \cdot \nabla \times \vec{F} = 0$$

∎

More product vector identities

$$\left\{\begin{array}{ll}
\text{xiii)} & \nabla\cdot(\vec F\times\vec G) = \vec G\cdot(\nabla\times\vec F) - \vec F\cdot(\nabla\times\vec G) \\
\text{xiv)} & \nabla\times(\vec F\times\vec G) = (\vec G\cdot\nabla)\vec F - (\vec F\cdot\nabla)\vec G + (\nabla\cdot\vec G)\vec F - (\nabla\cdot\vec F)\vec G \\
\text{xv)} & \nabla(\vec F\cdot\vec G) = (\vec F\cdot\nabla)\vec G + (\vec G\cdot\nabla)\vec F + \vec F\times(\nabla\times\vec G) + \vec G\times(\nabla\times\vec F) \\
\text{xvi)} & \nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f \\
\text{xvii)} & \nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f\cdot\nabla g
\end{array}\right.$$

eg Proof of xiv):

$$\begin{aligned}
[\nabla\times(\vec F\times\vec G)]_i &= \epsilon_{ijk}\partial_j(\vec F\times\vec G)_k = \epsilon_{ijk}\partial_j(\epsilon_{klm}F_l G_m) \\
&= \epsilon_{ijk}\epsilon_{lmk}\partial_j(F_l G_m) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(F_l\partial_j G_m + G_m\partial_j F_l) \\
&= F_i\partial_j G_j - F_j\partial_j G_i + G_j\partial_j F_i - G_i\partial_j F_j \\
&= (\nabla\cdot\vec G)F_i - (\vec F\cdot\nabla)G_i + (\vec G\cdot\nabla)F_i - (\nabla\cdot\vec F)G_i
\end{aligned}$$

☒

eg Proof of xv): It is easiest to begin from the r.h.s. and simplify:

$$\begin{aligned}
(\text{r.h.s.})_i &= (\vec F\cdot\nabla)G_i + (\vec G\cdot\nabla)F_i + (\vec F\times(\nabla\times\vec G))_i + (\vec G\times(\nabla\times\vec F))_i \\
&= F_j\partial_j G_i + G_j\partial_j F_i + \epsilon_{ijk}F_j\underbrace{(\nabla\times\vec G)_k}_{\epsilon_{klm}\partial_l G_m} + \epsilon_{ijk}G_j\underbrace{(\nabla\times\vec F)_k}_{\epsilon_{klm}\partial_l F_m} \\
&= F_j\partial_j G_i + G_j\partial_j F_i + \epsilon_{ijk}\epsilon_{lmk}F_j\partial_l G_m + \epsilon_{ijk}\epsilon_{lmk}G_j\partial_l F_m \\
&= F_j\partial_j G_i + G_j\partial_j F_i + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(F_j\partial_l G_m + G_j\partial_l F_m) \\
&= \cancel{F_j\partial_j G_i} + \cancel{G_j\partial_j F_i} + F_j\partial_i G_j - \cancel{F_j\partial_j G_i} + G_j\partial_i F_j - \cancel{G_j\partial_j F_i} \\
&= F_j\partial_i G_j + G_j\partial_i F_j \\
&= \partial_i(F_j G_j) = \partial_i(\vec F\cdot\vec G) = [\nabla(\vec F\cdot\vec G)]_i
\end{aligned}$$

☒

Laplacian - of a scalar field $f$ :

$$\text{Laplacian of } f = \Delta f = \nabla^2 f = \operatorname{div}(\operatorname{grad} f)$$

$$\underset{\text{Cartesian coordinates}}{=} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- a very important operator in applications
  eg in the description of equilibrium and diffusive processes.

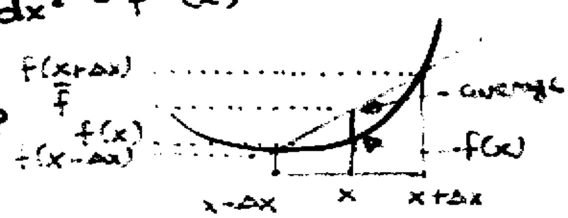
$$\boxed{\nabla^2 f = 0 \quad : \quad \text{Laplace's equation}}$$

- solutions of Laplace's equation are called
  harmonic functions

Interpretation of $\nabla^2 f$ :

One dimension : $f = f(x) \Rightarrow \nabla^2 f = \frac{d^2 f}{dx^2} = f''(x)$ measures convexity of $f$

$f'' > 0 \Rightarrow f$ is concave up




$\Rightarrow$ value of $f$ at $x$ lies below
the secant line connecting the values of $f$ at $x-\Delta x$, $x+\Delta x$

$$f(x) < \underbrace{\frac{f(x+\Delta x) + f(x-\Delta x)}{2}}_{\text{average of neighbours} \approx \text{local average } \bar{f} \text{ of } f} \approx \bar{f}(x)$$

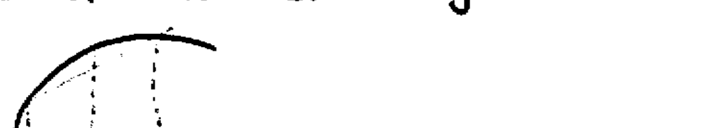
Recall $f''(x) = \lim_{\Delta x \to 0} \dfrac{\dfrac{f(x+\Delta x) - f(x)}{\Delta x} - \dfrac{f(x) - f(x-\Delta x)}{\Delta x}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} = \lim_{\Delta x \to 0} \frac{2}{\Delta x^2}\left[\frac{f(x+\Delta x) + f(x-\Delta x)}{2} - f(x)\right]$$

So $f''(x) > 0$ means that for small $\Delta x$, $\dfrac{f(x+\Delta x) + f(x-\Delta x)}{2} > f(x)$

Similarly, $f''(x) < 0 \Rightarrow f$ is greater than its local average

$$f(x) - \bar{f}(x) \approx -\frac{\Delta x^2}{2} f''(x)$$



Higher derivatives:

Laplacian $\nabla^2 f$ generalizes second derivative
- measures difference between $f$ at the point $\vec{x}$ and the "local average" $\bar{f}$ of $f$ near $\vec{x}$.

$$f(\vec{x}) - \bar{f} \doteq -M \nabla^2 f$$

---

Applications:

- Fluid dynamics — density $\rho$, velocity field $\vec{v}$

Mass conservation $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

Incompressible fluid : $\rho$ = constant $\Rightarrow \nabla \cdot \vec{v} = 0$

Irrotational fluid $\Rightarrow \nabla \times \vec{v} = \vec{0}$

- we will learn later that this implies:

$\Rightarrow$ there is a scalar field $\phi$ such that $\vec{v} = \nabla \phi$

$\phi$: velocity potential (recall $\nabla \times \nabla \phi = \vec{0}$)

Substitute $\vec{v} = \nabla \phi$ into $\nabla \cdot \vec{v} = 0$:

$$\operatorname{div}(\operatorname{grad} \phi) = \nabla \cdot (\nabla \phi) = 0 \Rightarrow \underline{\nabla^2 \phi = 0}$$

irrotational / potential flow : the velocity potential satisfies Laplace's equation

---

- Diffusive processes : heat transfer

$T(\vec{x}, t)$ : temperature field — $T(\vec{x},t)$: temperature of body at position $\vec{x}$, time $t$

$\vec{F}(\vec{x}, t)$ : heat flux density vector field

$\vec{F} \cdot \hat{n} \, \Delta S$ is heat flowing through surface $\Delta S$, normal $\hat{n}$ per unit time

Total outward heat flux through surface of a small region with volume $\Delta V$: $(\operatorname{div} \vec{F}) \Delta V$

<u>Fourier's Law</u> : $\vec{F} = -k \nabla T$ — heat flows from hot to cold

(good approximation)

$k>0$ : thermal conductivity of material

$\Rightarrow$ heat flux per unit volume $\nabla\cdot\vec{F} = -\nabla\cdot(k\nabla T) \underset{\substack{k \text{ constant} \\ \text{(uniform body)}}}{=} -k\nabla^2 T$

$\nabla^2 T < 0 \Rightarrow \nabla\cdot\vec{F} > 0$ : positive outward heat flux
$\Rightarrow$ temperature decreases

(temperature at $\vec{x}$ is greater than local average)

$\nabla^2 T > 0 \Rightarrow \nabla\cdot\vec{F} < 0$ : inward heat flux $\Rightarrow$ temperature increases

At <u>equilibrium</u>: no heat flux, temperature field satisfies

<u>Laplace's equation</u>: $\boxed{\nabla^2 T = 0.}$

In the presence of heat sources/sinks:

<u>Poisson's equation</u>: $\boxed{\nabla^2 T = \underbrace{g(\vec{x},t)}_{\text{source term}}}$

Time dependence:

Total heat content per unit volume $\underbrace{\rho}_{\substack{\text{density} \\ \text{mass/volume}}} \underbrace{c_p}_{\substack{\text{specific heat} \\ \text{heat (calories)/} \\ \text{(mass}\cdot\text{temperature)}}} \underbrace{T}_{\text{temperature}}$

(energy) measured in calories or Joules (or kJ)

Conservation of heat: rate of decrease of heat (per unit volume)

= net outward flux − source terms (external heating/cooling)

$$-\frac{\partial}{\partial t}(\rho c_p T) = \nabla\cdot\vec{F} - Q = -\nabla\cdot(k\nabla T) - Q$$

$\rho, c_p, k$ constant:

$$\Rightarrow \boxed{\frac{\partial T}{\partial t} = K\nabla^2 T + \frac{Q}{c_p\rho}}$$ : <u>heat equation</u> (with source)

$K = k/c_p\rho$ : thermal diffusivity

heat equation: $\frac{\partial T}{\partial t} = K\nabla^2 T$

<u>equilibrium</u>: $\frac{\partial T}{\partial t} = 0.$

—

General diffusive processes: $f(\vec{x},t)$ : concentration eg of chemical species

<u>Fick's Law</u> : flux proportional to concentration gradient $\vec{F} = -k\nabla f$

Conser<u>vation law</u> (no source): $\frac{\partial f}{\partial t} + \nabla\cdot\vec{F} = 0 \Rightarrow \boxed{\frac{\partial f}{\partial t} = k\nabla^2 f}$ Diffusion equation

Equilibrium of diffusive process: $\boxed{\nabla^2 f = 0}$ Laplace's equation

# Change of Coordinates

· Linear Orthogonal Transformations (Cartesian coordinates)

basis vectors $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ constant, independent of $\vec{R}$

Recall:

**Coordinates:** $\vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$

project onto $\hat{\imath} = \hat{e}_1$:

$$x = \vec{R}\cdot\hat{\imath} = x'\,\hat{\imath}\cdot\hat{\imath}' + y'\,\hat{\imath}\cdot\hat{\jmath}' + z'\,\hat{\imath}\cdot\hat{k}'$$

$$= J_{11}x' + J_{12}y' + J_{13}z' \qquad \text{etc}$$

Write $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, $\{x_1, x_2, x_3\} = \{x, y, z\}$;

in this notation:

$$x_1 = J_{11}x_1' + J_{12}x_2' + J_{13}x_3' = J_{1j}x_j' \quad \left(\begin{matrix}\text{summation}\\ \text{convention}\end{matrix}\right)$$

$$\text{ie}\quad x_i = J_{ij}x_j', \qquad J_{ij} = \hat{e}_i\cdot\hat{e}_j' = \frac{\partial x_i}{\partial x_j'}$$

["old" coordinates in terms of "new" coordinates: $\begin{pmatrix}x\\y\\z\end{pmatrix} = J\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$ "New" from "old": $\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = J^T\begin{pmatrix}x\\y\\z\end{pmatrix}$

Recall $J$ is orthogonal, $J^T = J^{-1}$, and $J = \dfrac{\partial(x,y,z)}{\partial(x',y',z')} = \dfrac{\partial(x_1,x_2,x_3)}{\partial(x_1',x_2',x_3')}$]

$\Rightarrow J_{ij}J_{kj} = \delta_{ik}$

**Components of vectors:** $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k} = A_1'\hat{\imath}' + A_2'\hat{\jmath}' + A_3'\hat{k}'$

$$\Rightarrow \vec{A}\cdot\hat{\imath} = A_1 = J_{11}A_1' + J_{12}A_2' + J_{13}A_2' \qquad \text{etc}$$

(or in tensor notation $\vec{A} = A_i\hat{e}_i = A_i'\hat{e}_i'$

and $A_i = J_{ij}A_j'$, $A_j' = (J^T)_{ji}A_i = J_{ij}A_i$ )

**Transformation of scalar field $f$** $= f(\vec{x}) = f(\vec{R})$:

The value of $f$ at the position $\vec{x} = \vec{R}$ does not depend on the coordinate system used, but the formula for $f$ in terms of the coordinates will be different.

In "old" coordinates $f(x,y,z)$

In "new" coordinates $f'(x',y',z') = f(x(x',y',z'), y(x',y',z'), z(x',y',z'))$

Transformation of vector field $\vec{F} = \vec{F}(\vec{x})$ :

Two steps: • express components of $\vec{F}$ w.r.t. new basis in terms of components w.r.t. old basis ($F_i$) $\nearrow F_i'$

• write "old" components $F_i$ in terms of new coordinates

$$\begin{pmatrix} F_1' \\ F_2' \\ F_3' \end{pmatrix}(x',y',z') = J^T \begin{pmatrix} F_1\,(x(x',y',z'),\, y(x',y',z'),\, z(x',y',z')) \\ F_2\,(x(x',y',z'),\, y(x',y',z'),\, z(x',y',z')) \\ F_3\,(x(x',y',z'),\, y(x',y',z'),\, z(x',y',z')) \end{pmatrix}$$

---

For a given scalar field $f$, the vector field grad $f$ computed in the "old" and "new" coordinate systems represents the same vector field.

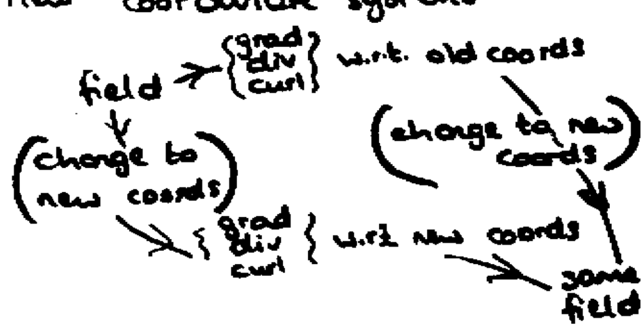
"old": $\text{grad}\, f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = \frac{\partial f}{\partial x_i}\hat{e}_i$ ← derivatives w.r.t. old coordinates

"new": $\text{grad}' f = \frac{\partial f}{\partial x_i'}\hat{e}_i'$ ← derivatives w.r.t. new coordinates

(if we compute the components of grad $f$ ("old" derivatives) w.r.t. the new basis $\{\hat{e}_i'\}$, we get $\tilde{f}_i' = (J^T)_{ij}\frac{\partial f}{\partial x_j} = J_{ji}\frac{\partial f}{\partial x_j}$, which are exactly the components of $\text{grad}' f$ ← "new" derivatives

– by the chain rule: $\frac{\partial f}{\partial x_i'} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = J_{ji}\frac{\partial f}{\partial x_j}$ )

Similarly, for a given vector field $\vec{G}$, computation of div $\vec{G}$ and curl $\vec{G}$ in the "old" or "new" coordinate systems represents the same field.




Reason:

grad, div, curl
are vector operations
which can be defined geometrically (intrinsically),
without reference to a coordinate system

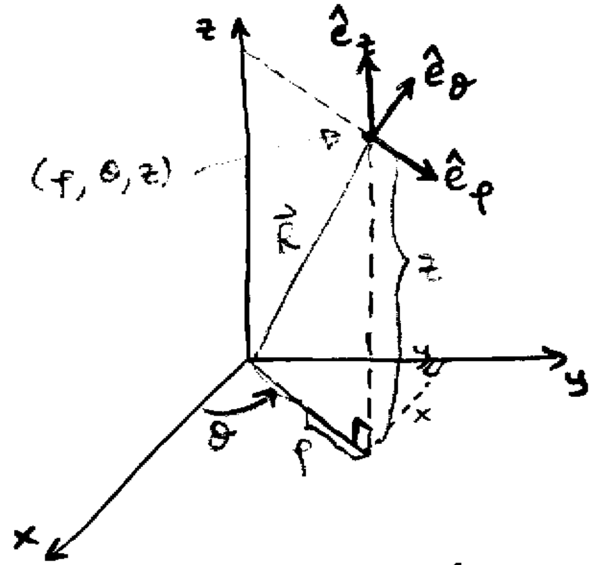
• grad $f$ : magnitude – maximum rate of increase of $f$ with distance, direction of greatest increase
• div $\vec{F}$ : flux per unit volume
• curl $\vec{F}$ : circulation per unit area / twice local angular velocity

# Cylindrical Coordinates

$$(x, y, z) \longrightarrow (\rho, \theta, z)$$

Notation:
often: $r$ instead of $\rho$
sometimes: $\phi$ instead of $\theta$

$z$ : height of point above x-y plane
$\rho, \theta$ : polar coordinates of projection of point onto x-y plane




$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

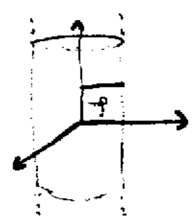
$\iff$

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \tan^{-1} y/x = \sin^{-1}\dfrac{y}{\sqrt{x^2+y^2}} \\ z = z \end{cases}$$

(Note: on the z-axis $\rho = 0$, $\theta$ is not defined)

## Level surfaces: (think of $\rho, \theta, z$ as scalar fields)

- $\rho$ = constant: cylinders

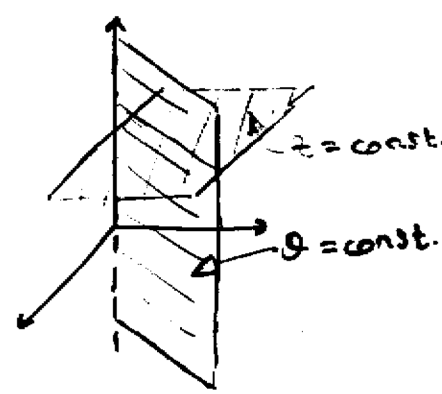



(thus "cylindrical coordinates")

grad $\rho = \nabla\rho$ normal to this level surface
(points away from z-axis)

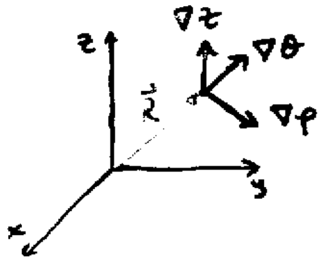
- $\theta$ = constant : vertical half-plane, extending out from z-axis

Normal $\nabla\theta$ (counterclockwise)




- $z$ = constant : horizontal plane

Normal $\nabla z$ (points up)




The surfaces $\rho$ = constant, $\theta$ = constant, $z$ = constant intersect everywhere at right angles

$\Rightarrow$ $\nabla\rho$, $\nabla\theta$, $\nabla z$ are mutually perpendicular, form a right-handed coordinate system

## Coordinate curves

- The intersection of any two level surfaces $\theta$ = constant, $z$ = constant gives a curve along which only $\rho$ varies (horizontal ray extending from $z$-axis)

  - a coordinate curve for $\rho$ ← crosses level surfaces of $\rho$ orthogonally

  Tangent: grad $\rho = \nabla\rho$

  Unit vector in direction of grad $\rho$: $\hat{e}_\rho = \dfrac{\nabla\rho}{|\nabla\rho|}$

  direction of increasing $\rho$

  (unit tangent to coordinate curve)

- Intersection of $\rho$ = constant and $z$ = constant:

  coordinate curve for $\theta$ (horizontal circles) centred on $z$-axis

  Tangent $\nabla\theta$, unit tangent $\hat{e}_\theta = \dfrac{\nabla\theta}{|\nabla\theta|}$

- Intersection of $\rho$ = constant and $\theta$ = constant:

  coordinate curve for $z$ (vertical lines)

  Tangent $\nabla z$, unit tangent $\hat{e}_z = \dfrac{\nabla z}{|\nabla z|} = \hat{k}$

---

Position vector of an arbitrary point

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \leftarrow \text{Cartesian}$$

$$= \rho\,\hat{e}_\rho + z\,\hat{e}_z \qquad \leftarrow \text{cylindrical}$$

Scalar field $f$: $\quad f(\rho, \theta, z)$

Vector field $\vec{F}$: $\quad \vec{F} = F_\rho\,\hat{e}_\rho + F_\theta\,\hat{e}_\theta + F_z\,\hat{e}_z$

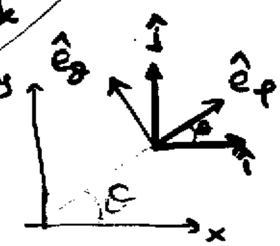
Expressions for vector differential operators (grad, div, curl, Laplacian) in cylindrical coordinates:

**Method 1:** Change of coordinates from expression in Cartesian coordinates using change of basis and chain rule

("long and tedious calculation, little insight")

eg gradient:

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$




Coordinates:
$$\left.\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= z \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1} y/x \\ z &= z \end{aligned}\right.$$

Basis vectors:
$$\left.\begin{aligned} \hat{e}_\rho &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ \hat{e}_\theta &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \\ \hat{e}_z &= \hat{k} \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} \hat{i} &= \cos\theta\,\hat{e}_\rho - \sin\theta\,\hat{e}_\theta \\ \hat{j} &= \sin\theta\,\hat{e}_\rho + \cos\theta\,\hat{e}_\theta \\ \hat{k} &= \hat{e}_z \end{aligned}\right.$$

Partial derivatives:
$$\frac{\partial\rho}{\partial x} = \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\rho} = \cos\theta,$$

$$\frac{\partial\rho}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{\rho} = \sin\theta, \quad \frac{\partial\rho}{\partial z} = 0$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = -\frac{y}{\rho^2} = -\frac{\sin\theta}{\rho},$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+y^2/x^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{\rho^2} = \frac{\cos\theta}{\rho}, \quad \frac{\partial\theta}{\partial z} = 0$$

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial z} = 1$$

---

By the chain rule:
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$
$$= \cos\theta\frac{\partial f}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}$$
$$\frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial f}{\partial\theta}, \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}$$

---

Combining everything:
$$\text{grad} f = \left(\cos\theta\frac{\partial f}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}\right)(\cos\theta\,\hat{e}_\rho - \sin\theta\,\hat{e}_\theta)$$
$$+ \left(\sin\theta\frac{\partial f}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial f}{\partial\theta}\right)(\sin\theta\,\hat{e}_\rho + \cos\theta\,\hat{e}_\theta) + \frac{\partial f}{\partial z}\hat{e}_z$$

simplifying
$$\Rightarrow \text{grad} f = \nabla f = \frac{\partial f}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$$

similarly for div, curl...

**Method 2**: Geometrical interpretation of grad, div, curl

Recall: $|\text{grad } f| = \left|\frac{df}{ds}\right|$ where $s$ measures distance in the direction of grad $f$ (maximum value of directional derivative

Distance along coordinate curves:

Along coordinate curve of $z$: $ds = |dz|$ $\qquad \frac{\partial \vec{R}}{\partial z} = \hat{e}_z$

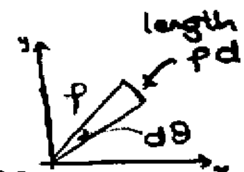
$$\Rightarrow |\text{grad } z| = \left|\frac{dz}{ds}\right|_{\rho,\theta \text{ constant}} = 1$$

$$\Rightarrow \boxed{\hat{e}_z = \nabla z} \quad (= \hat{z})$$

Along coordinate curve of $\rho$: $ds = |d\rho|$ $\qquad \frac{\partial \vec{R}}{\partial \rho} = \hat{e}_\rho$

$$\Rightarrow |\text{grad } \rho| = \left|\frac{d\rho}{ds}\right|_{\theta, z \text{ constant}} = 1$$

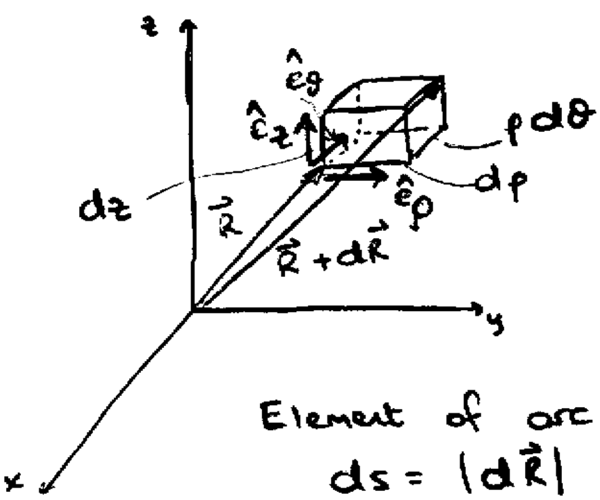
$$\Rightarrow \boxed{\hat{e}_\rho = \nabla\rho} \quad (= \hat{\rho})$$




Along coordinate curve of $\theta$: $ds = \rho|d\theta|$

$$\Rightarrow |\text{grad } \theta| = \left|\frac{d\theta}{ds}\right|_{\rho, z \text{ constant}} = \left|\frac{d\theta}{\rho d\theta}\right| = 1/\rho$$

$$\Rightarrow \boxed{\hat{e}_\theta = \rho\nabla\theta} \quad (= \hat{\theta}) \qquad \frac{\partial \vec{R}}{\partial \theta} = \rho\,\hat{e}_\theta$$




Displacement from $\vec{R}$ (coordinates $(\rho, \theta, z)$) to $\vec{R} + d\vec{R}$ ( $(\rho+d\rho, \theta+d\theta, z+dz)$ )

$$d\vec{R} = d\rho\,\hat{e}_\rho + \rho d\theta\,\hat{e}_\theta + dz\,\hat{e}_z$$
$$= \frac{\partial \vec{R}}{\partial \rho} d\rho + \frac{\partial \vec{R}}{\partial \theta} d\theta + \frac{\partial \vec{R}}{\partial z} dz$$

Element of arc length:

$$ds = |d\vec{R}| = \left(d\rho^2 + \rho^2 d\theta^2 + dz^2\right)^{1/2}$$

$$\boxed{ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2}$$

Volume element

$$\boxed{dV = d\rho \cdot \rho d\theta \cdot dz = \rho\, d\rho\, d\theta\, dz}$$

Jacobian

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \rho$$

- <u>Gradient</u> grad $f = \nabla f$ in cylindrical coordinates

Consider the scalar field $f$: $f(\rho, \theta, z)$ (a scalar function of position)

Recall: $\hat{e}_\rho, \hat{e}_\theta, \hat{e}_z$ are mutually orthogonal unit vectors $\Rightarrow$ orthonormal basis

Thus:

$$\nabla f = \underbrace{(\hat{e}_\rho \cdot \nabla f)}_{\text{projection of } \nabla f \text{ onto } \hat{e}_\rho} \hat{e}_\rho + (\hat{e}_\theta \cdot \nabla f)\hat{e}_\theta + (\hat{e}_z \cdot \nabla f)\hat{e}_z$$

$$\hat{e}_\rho \cdot \nabla f = D_{\hat{e}_\rho} f \qquad \text{(directional derivative of } f \text{ in direction } \hat{e}_\rho)$$

$$= \left.\frac{df}{ds}\right|_{\theta, z \text{ constant}} = \frac{\partial f}{\partial \rho}$$

Similarly

$$\hat{e}_\theta \cdot \nabla f = D_{\hat{e}_\theta} f = \left.\frac{df}{ds}\right|_{\rho, z \text{ const}} = \frac{1}{\rho}\frac{\partial f}{\partial \theta}$$

$$\hat{e}_z \cdot \nabla f = D_{\hat{e}_z} f = \left.\frac{df}{ds}\right|_{\rho, \theta \text{ const}} = \frac{\partial f}{\partial z}$$

$\Rightarrow$ Gradient $\nabla f$ in cylindrical coordinates

$$\boxed{\text{grad } f = \nabla f = \frac{\partial f}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z}$$

---

- <u>Divergence</u> div $\vec{F} = \nabla \cdot \vec{F}$ in cylindrical coordinates

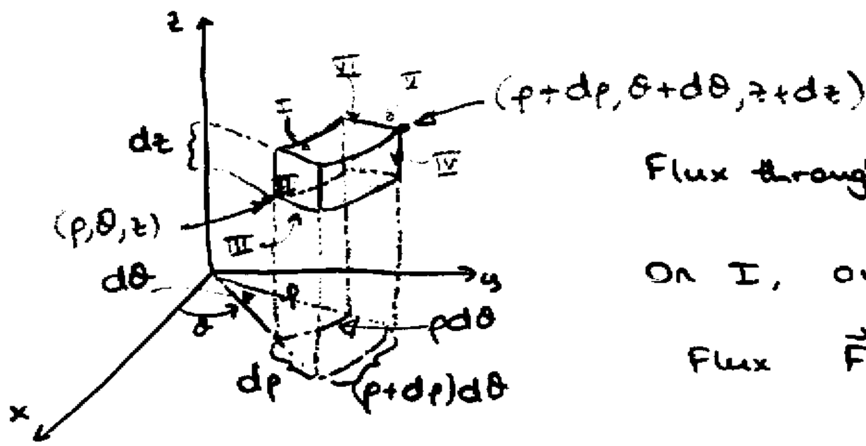
Consider the vector field $\vec{F}$ ← flux density

$$\vec{F} = F_\rho(\rho, \theta, z)\hat{e}_\rho + F_\theta(\rho, \theta, z)\hat{e}_\theta + F_z(\rho, \theta, z)\hat{e}_z$$

<u>Divergence</u>: Flux per unit volume

Consider a small volume $\Delta V \approx dV = \rho \, d\rho \, d\theta \, dz$

Flux through a surface with normal $\hat{n}$, area $dS$ is $\vec{F} \cdot \hat{n} \, dS$




Flux through surfaces I and IV:

On I, outward normal $-\hat{e}_\rho$, area $\rho\, d\theta\, dz$

Flux $\vec{F}\cdot\hat{n}\, dS\big|_{I} = -F_\rho(\rho,\theta,z)\,\rho\, d\theta\, dz$

On IV, outward normal $+\hat{e}_\rho$, area $(\rho+d\rho)\,d\theta\, dz$

Face I ($\rho$ = constant)


Flux $\vec{F}\cdot\hat{n}\, dS\big|_{IV} = +F_\rho(\rho+d\rho,\theta,z)(\rho+d\rho)\,d\theta\, dz$

Net contribution from I & IV:

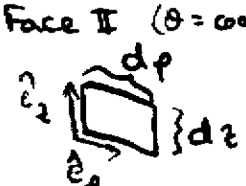
$$\rho F_\rho\, d\theta\, dz\Big|_{IV} - \rho F_\rho\, d\theta\, dz\Big|_{I} = \frac{\left[(\rho+d\rho)F_\rho(\rho+d\rho,\theta,z) - \rho F_\rho(\rho,\theta,z)\right]}{d\rho}\, d\rho\, d\theta\, dz$$

(limit $d\rho\to 0$)

$$= \frac{\partial(\rho F_\rho)}{\partial\rho}\, d\rho\, d\theta\, dz$$

Flux through faces II and V: on II, $\hat{n} = -\hat{e}_\theta$ ; on V, $\hat{n} = \hat{e}_\theta$

Face II ($\theta$ = const.)




$$F_\theta\, d\rho\, dz\Big|_{V} - F_\theta\, d\rho\, dz\Big|_{II} = \frac{\partial F_\theta}{\partial\theta}\, d\rho\, d\theta\, dz$$

Flux through faces III and VI: on III, $\hat{n} = -\hat{e}_z$ ; on VI, $\hat{n} = \hat{e}_z$

Face III ($z$ = const.)


$$F_z\, \rho\, d\rho\, d\theta\Big|_{VI} - F_z\, \rho\, d\rho\, d\theta\Big|_{III} = \frac{\partial F_z}{\partial z}\,\bar{\rho}\, d\rho\, d\theta\, dz$$

($\rho \le \bar{\rho} \le \rho + d\rho$, $\bar{\rho}\to\rho$ as $dV\to 0$)

Total flux: $\left(\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{\partial F_\theta}{\partial\theta} + \bar{\rho}\frac{\partial F_z}{\partial z}\right) d\rho\, d\theta\, dz$

Volume $dV = \rho\, d\rho\, d\theta\, dz$

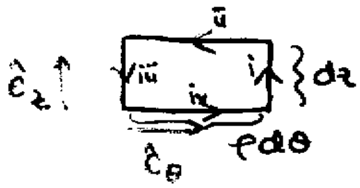
$\Rightarrow$ divergence $\nabla\cdot\vec{F}$ in cylindrical coordinates

$$\boxed{\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}}$$

• Curl $\text{curl}\,\vec{F} = \nabla\times\vec{F}$ in cylindrical coordinates

Curl : Circulation per unit area $\left[\lim_{S\to 0}\frac{1}{S}\oint \vec{F}\cdot\vec{T}\,d\ell\right]$

$\hat{e}_\rho$ component – compute circulation around a surface element with normal $\hat{e}_\rho$ : Circulation about face I




Along sides i and iii :
unit tangent $\hat{e}_z$ $-\hat{e}_z$

$$\vec{F}\cdot\vec{T}\,d\ell = F_z\,dz\Big|_{\substack{i\\(\rho,\theta+d\theta,z)}} - F_z\,dz\Big|_{\substack{iii\\(\rho,\theta,z)}} = \frac{\partial F_z}{\partial\theta}\,d\theta\,dz$$

Along sides ii and iv : $-\hat{e}_\theta$ $\hat{e}_\theta$

$$\vec{F}\cdot\vec{T}\,d\ell = -F_\theta\,\rho\,d\theta\Big|_{\substack{ii\\(\rho,\theta,z+dz)}} + F_\theta\,\rho\,d\theta\Big|_{\substack{iv\\(\rho,\theta,z)}} = -\frac{\partial}{\partial z}(\rho F_\theta)\,d\theta\,dz$$
$$= -\rho\frac{\partial F_\theta}{\partial z}\,d\theta\,dz$$

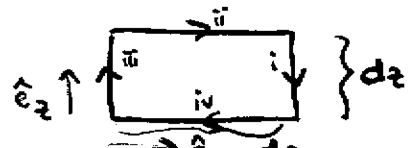
Area $= \rho\,d\theta\,dz$

$\hat{e}_\rho$ component of curl $\vec{F}$ is total circulation ("swirl") divided by area :

$$\Rightarrow \hat{e}_\rho\cdot\text{curl}\,\vec{F} = \frac{1}{\rho}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}$$

---

$\hat{e}_\theta$ component




Circulation about face II, normal $\hat{e}_\theta$

($\hat{e}_\theta$ points into page: take circulation clockwise)

Along sides i and iii : $-\hat{e}_z$ $\hat{e}_z$ $\vec{F}\cdot\vec{T}\,d\ell = -F_z\,dz\big|_i + F_z\,dz\big|_{iii} = -\dfrac{\partial F_z}{\partial\rho}\,d\rho\,dz$

Along sides ii and iv : $\hat{e}_\rho$ $-\hat{e}_\rho$ $\vec{F}\cdot\vec{T}\,d\ell = F_\rho\,d\rho\big|_{ii} - F_\rho\,d\rho\big|_{iv} = \dfrac{\partial F_\rho}{\partial z}\,d\rho\,dz$

Area $d\rho\,dz$ $\Rightarrow \hat{e}_\theta\cdot\text{curl}\,\vec{F} = \dfrac{\partial F_\rho}{\partial z} - \dfrac{\partial F_z}{\partial\rho}$

---

$\hat{e}_z$ component


Circulation about face III, normal $\hat{e}_z$

Along sides i and iii: $\hat{e}_\theta$, $-\hat{e}_\theta$: $\vec{F}\cdot\vec{T}\,d\ell = \rho F_\theta\, d\theta\big|_{i} - \rho F_\theta\, d\theta\big|_{iii} = \frac{\partial}{\partial\rho}(\rho F_\theta)\, d\rho\, d\theta$ $(\rho+d\rho,\theta,z)$

Along sides ii and iv: $\vec{F}\cdot\vec{T}\,d\ell = -F_\rho\, d\rho\big|_{ii} + F_\rho\, d\rho\big|_{iv} = -\frac{\partial F_\rho}{\partial\theta}\, d\rho\, d\theta$

Area $\rho\, d\rho\, d\theta$ $\Rightarrow$ $\hat{e}_z\cdot \text{curl}\,\vec{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\theta) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial\theta}$

$\Rightarrow$ curl $\nabla\times\vec{F}$ in cylindrical coordinates

$$\boxed{\text{curl}\,\vec{F} = \nabla\times\vec{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\hat{e}_\theta + \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\theta) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial\theta}\right)\hat{e}_z}$$

$$= \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

---

- Laplacian of scalar field in cylindrical coordinates

$$\nabla^2 f = \text{div grad}\, f = \nabla\cdot\left(\frac{\partial f}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z\right)$$

$$\Rightarrow \boxed{\nabla^2 f = \underbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)}_{\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho}} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}}$$

- Laplacian of vector field:

Define $\nabla^2\vec{F}$ using the vector identity

$$\nabla^2\vec{F} = \nabla(\nabla\cdot\vec{F}) - \nabla\times(\nabla\times\vec{F}) = \text{grad}(\text{div}\,\vec{F}) - \text{curl}(\text{curl}\,\vec{F})$$

---

Note: all formulas in 2-d polar coordinates (grad, div, scalar curl, Laplacian) can be found from the formulas in cylindrical coordinates using $F_z = 0$, $\hat{e}_z = \vec{0}$, $\frac{\partial}{\partial z} = 0$ (no $z$ dependence)

# Spherical Coordinates

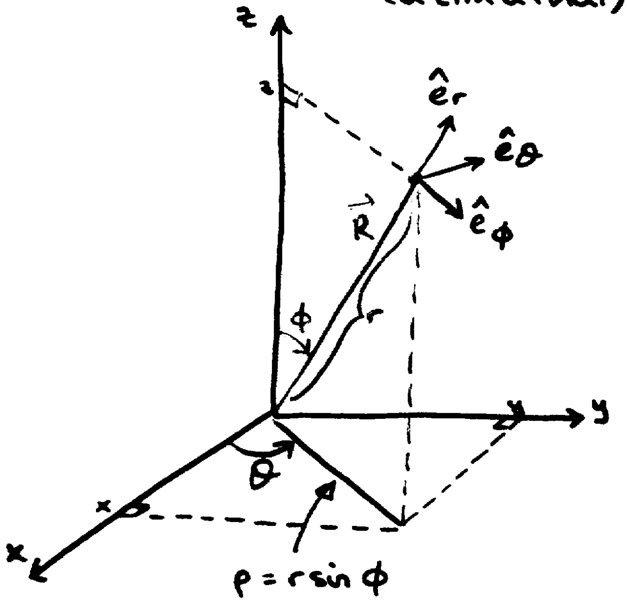
$(x, y, z) \to (r, \phi, \theta)$

[Notational convention: Mathematics: $(r, \phi, \theta)$; Physics: $(r, \theta, \phi)$]

$r$ : radial distance of point from origin $= |\vec{R}|$

$\phi$ : angle of $\vec{R}$ from positive $z$-axis $\quad 0 \le \phi \le \pi$ "co-latitude"

$\theta$ : angle of projection onto $x$-$y$ plane from positive $x$-axis (azimuthal) $\quad 0 \le \theta < 2\pi$ "longitude"




$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases}$$

$\Leftrightarrow$

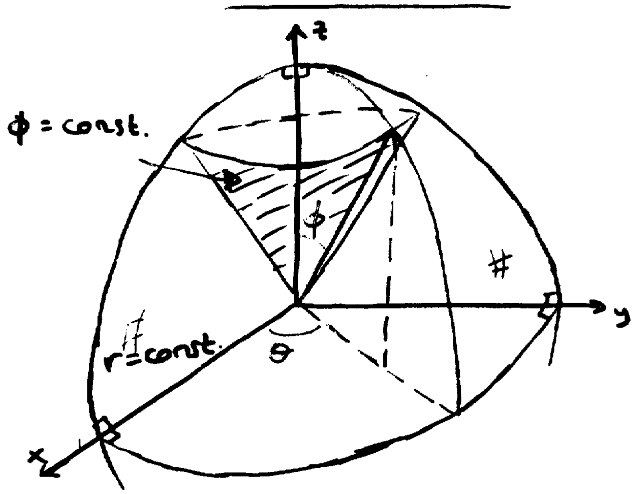
$$\begin{cases} r = \sqrt{x^2+y^2+z^2} = |\vec{R}| \\ \phi = \cos^{-1}\dfrac{z}{\sqrt{x^2+y^2+z^2}} \quad \text{(principal value) } 0 \le \phi \le \pi \\ \theta = \tan^{-1} y/x = \sin^{-1}\dfrac{y}{\sqrt{x^2+y^2}} \end{cases}$$

[Relation to cylindrical coordinates: $\theta = \theta$; $\left.\begin{matrix}\rho = r\sin\phi \\ z = r\cos\phi\end{matrix}\right\} \Leftrightarrow \begin{cases} r = \sqrt{\rho^2+z^2} \\ \phi = \tan^{-1}\rho/z \end{cases}$]

Consider the Earth, regarded as a sphere $r =$ constant: (latitude $= \pi - \phi$ in degrees)

$\phi = 0$ : North Pole; $\phi = \pi/2$ : Equator; $\phi = \pi$ : South Pole

## Level surfaces:




- $r$ = constant : spheres ("spherical coordinates")

  Normal $\nabla r$ : (points in local vertical direction, radial)

- $\phi$ = constant : cones

  Normal $\nabla\phi$ (points due South)

- $\theta$ = constant : vertical half-planes

  Normal $\nabla\theta$ (points due East, counterclockwise)

## Coordinate curves

- $\phi$ = constant, $\theta$ = constant : coordinate curve for $r$
  (ray directed away from origin)

Tangent $\nabla r$, unit tangent $\hat{e}_r = \dfrac{\nabla r}{|\nabla r|}$

Along the coordinate curve of $r$, $\boxed{ds = |dr|}$

$$\Rightarrow |\nabla r| = \left|\frac{dr}{ds}\right|_{\theta,\phi \text{ constant}} = \left|\frac{dr}{dr}\right| = 1 \quad \Rightarrow \quad \hat{e}_r = \nabla r$$

- $r$ = constant, $\theta$ = constant : coordinate curve for $\phi$
  ("line of longitude", from North Pole to South Pole)
  - semicircle of constant longitude $\therefore$ radius $r$

Tangent $\nabla\phi$, unit tangent $\hat{e}_\phi = \dfrac{\nabla\phi}{|\nabla\phi|}$

Along these semicircles, $\boxed{ds = r|d\phi|}$ $\Rightarrow |\nabla\phi| = \dfrac{1}{r}$

$$\hat{e}_\phi = r\nabla\phi$$

- $r$ = constant, $\phi$ = constant : coordinate curve for $\theta$
  ("line of latitude") - circle of constant latitude
  radius $r\sin\phi$

Tangent $\nabla\theta$, unit tangent $\hat{e}_\theta = \dfrac{\nabla\theta}{|\nabla\theta|}$

Along these circles $\boxed{ds = r\sin\phi\,|d\theta|}$ $\Rightarrow |\nabla\theta| = \dfrac{1}{r\sin\phi}$

$$\hat{e}_\theta = r\sin\phi\,\nabla\theta$$

---

$\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_\theta$ form a right-handed coordinate system:

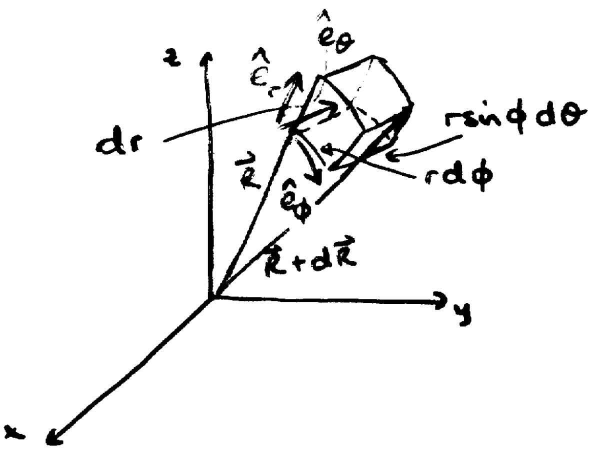
$$\hat{e}_r \times \hat{e}_\phi = \hat{e}_\theta$$

Position vector of an arbitrary point

$$\vec{R} = r\hat{e}_r \qquad (= x\hat{i} + y\hat{j} + z\hat{k})$$

Displacement from $\vec{R} : (r, \phi, \theta)$ to $\vec{R}+d\vec{R} : (r+dr, \phi+d\phi, \theta+d\theta)$

$$d\vec{R} = dr\,\hat{e}_r + r\,d\phi\,\hat{e}_\phi + r\sin\phi\,d\theta\,\hat{e}_\theta$$




Note: $d\vec{R} = \frac{\partial \vec{R}}{\partial r}dr + \frac{\partial \vec{R}}{\partial \phi}d\phi + \frac{\partial \vec{R}}{\partial \theta}d\theta$

$$\Rightarrow \frac{\partial \vec{R}}{\partial r} = \hat{e}_r, \quad \frac{\partial \vec{R}}{\partial \phi} = r\hat{e}_\phi, \quad \frac{\partial \vec{R}}{\partial \theta} = r\sin\phi\,\hat{e}_\theta$$

$$\left|\frac{\partial \vec{R}}{\partial r}\right| = 1, \quad \left|\frac{\partial \vec{R}}{\partial \phi}\right| = r = \frac{1}{|\nabla\phi|}, \quad \left|\frac{\partial \vec{R}}{\partial \theta}\right| = r\sin\phi = 1/|\nabla\theta|$$

Element of arc length:

$$ds = |d\vec{R}| = (dr^2 + r^2 d\phi^2 + r^2\sin^2\phi\, d\theta^2)^{1/2}$$

$$\boxed{ds^2 = dr^2 + r^2 d\phi^2 + r^2\sin^2\phi\, d\theta^2}$$

Volume element

$$dV = (dr)(r\,d\phi)(r\sin\phi\,d\theta) \Rightarrow \boxed{dV = \underbrace{r^2\sin\phi}_{\text{Jacobian } \frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}}\,dr\,d\phi\,d\theta}$$

The differential operators grad, div, curl, and Laplacian may be derived as for cylindrical coordinates; we will obtain the formulas as special cases of those for general orthogonal curvilinear coordinates.

# Orthogonal Curvilinear Coordinates

- Each point P (in some region of space) is uniquely represented by coordinates $(u_1, u_2, u_3)$ — possibly lengths, angles, ...

$$\left.\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned}\right\} \iff \left\{\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned}\right.$$

one-to-one correspondence

where we assume that all functions $x(u_1, u_2, u_3), \ldots$ and the inverse functions $u_1(x, y, z), \ldots$ are $C^1$ (continuously differentiable) functions.

eg for cylindrical coordinates $u_1 = \rho,\ u_2 = \theta,\ u_3 = z$

for spherical coordinates $u_1 = r,\ u_2 = \phi,\ u_3 = \theta$

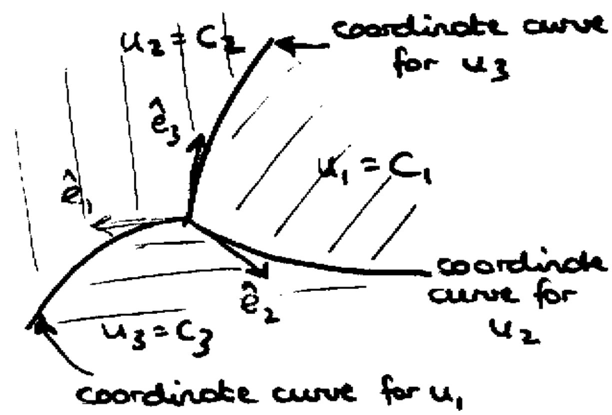
## Level surfaces

Any point P with coordinates $(c_1, c_2, c_3)$ lies on the intersection of three level surfaces

$u_1(x, y, z) = c_1$

$u_2(x, y, z) = c_2$

$u_3(x, y, z) = c_3$




Normal to the surface $u_i(x, y, z) = c_i$ is

$$\nabla u_i = \frac{\partial u_i}{\partial x}\hat{i} + \frac{\partial u_i}{\partial y}\hat{j} + \frac{\partial u_i}{\partial z}\hat{k}$$

Unit normal: $\hat{e}_i = \dfrac{\nabla u_i}{|\nabla u_i|}$

If the vectors $\nabla u_1, \nabla u_2, \nabla u_3$ are mutually orthogonal at every point, we say $(u_1, u_2, u_3)$ form orthogonal curvilinear coordinates.

Orthogonal curvilinear coordinates : $\nabla u_1, \nabla u_2, \nabla u_3$ are orthogonal $\Rightarrow \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

Assume $\nabla u_1, \nabla u_2, \nabla u_3$ (in order) form a right-handed system $\Rightarrow \hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

$(\Leftrightarrow \nabla u_1 \cdot (\nabla u_2 \times \nabla u_3) > 0 )$

Coordinate curves

The intersection of two level surfaces gives a curve along which only one of the coordinates varies: a coordinate curve.

Along the coordinate curve for $u_i$, $\vec{R}$ ← position vector depends only on $u_i$ (lies on level sets of $u_j$, $j \neq i$)

$\Rightarrow$ tangent to the coordinate curve $\dfrac{\partial \vec{R}}{\partial u_i} = \dfrac{\partial x}{\partial u_i}\hat{\imath} + \dfrac{\partial y}{\partial u_i}\hat{\jmath} + \dfrac{\partial z}{\partial u_i}\hat{k}$

- For orthogonal curvilinear coordinates, the normal $\nabla u_i$ to the level surface $u_i = \text{constant}$ is parallel to the tangent vector $\dfrac{\partial \vec{R}}{\partial u_i}$ of the corresponding coordinate curve

eg for $u_1$: Coordinate curve lies on $u_2 = \text{const}$, $u_3 = \text{const}$

$\Rightarrow$ tangent $\dfrac{\partial \vec{R}}{\partial u_1}$ is orthogonal to surface normals $\nabla u_2, \nabla u_3$

For orthogonal curvilinear coordinates, $\nabla u_1$ is $\perp$ $\nabla u_2, \nabla u_3$

$\Rightarrow \dfrac{\partial \vec{R}}{\partial u_1}$ is parallel to $\nabla u_1$

(not antiparallel, since both vectors point in the direction of increasing $u_1$)

Hence $\dfrac{\partial \vec{R}}{\partial u_1}, \dfrac{\partial \vec{R}}{\partial u_2}, \dfrac{\partial \vec{R}}{\partial u_3}$ form a right-handed system of mutually orthogonal vectors.

Unit vectors (orthonormal) : $\hat{e}_i = \dfrac{\nabla u_i}{|\nabla u_i|} = \dfrac{\partial \vec{R}/\partial u_i}{|\partial \vec{R}/\partial u_i|}$, $i = 1, 2, 3$

## Scale factor

Scale factor $h_i$ : rate at which arc length increases along the $i^{th}$ coordinate curve, w.r.t. $u_i$

$s_i$ : arc length along $i^{th}$ coordinate curve ← only $u_i$ varies (measured in direction of increasing $u_i$)

$$\Rightarrow \quad h_i = \frac{ds_i}{du_i}, \quad i = 1,2,3 \qquad \text{ie.} \quad \boxed{ds_i = h_i\, du_i}$$

We have

$$ds = |d\vec{R}| = \left| \frac{\partial \vec{R}}{\partial u_1} du_1 + \frac{\partial \vec{R}}{\partial u_2} du_2 + \frac{\partial \vec{R}}{\partial u_3} du_3 \right|$$

$$\Rightarrow ds_i = \left| \frac{\partial \vec{R}}{\partial u_i} du_i \right| \qquad \Rightarrow \quad \boxed{h_i = \left| \frac{\partial \vec{R}}{\partial u_i} \right|}$$

$du_j = 0$ for $j \neq i$

and $\frac{\partial \vec{R}}{\partial u_i} = h_i \hat{e}_i$

Alternative formula:

$|\nabla u_i|$ : rate of change of $u_i$ w.r.t. distance, in direction $\nabla u_i$

is in direction of coordinate curve for $u_i$

$s_i$ : distance along coordinate curve for $u_i$ ie // $\nabla u_i$

$$\Rightarrow |\nabla u_i| = \left| \frac{du_i}{ds_i} \right| = \frac{1}{h_i} \qquad \Rightarrow \quad \boxed{h_i = \frac{1}{|\nabla u_i|}}$$

Note: $$\nabla u_i \cdot \frac{\partial \vec{R}}{\partial u_j} = \frac{\partial u_i}{\partial x}\frac{\partial x}{\partial u_j} + \frac{\partial u_i}{\partial y}\frac{\partial y}{\partial u_j} + \frac{\partial u_i}{\partial z}\frac{\partial z}{\partial u_j}$$

$$= \frac{\partial u_i}{\partial u_j} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and $\nabla u_i$, $\frac{\partial \vec{R}}{\partial u_i}$ are parallel $\Rightarrow$ $\underbrace{|\nabla u_i|}_{1/h_i} \underbrace{\left| \frac{\partial \vec{R}}{\partial u_i} \right|}_{h_i} = 1$

eg cylindrical coordinates: $h_\rho = 1$, $h_\theta = \rho$, $h_z = 1$

spherical coordinates: $h_r = 1$, $h_\phi = r$, $h_\theta = r \sin\phi$

Element of arc length $ds = |d\vec{R}| = |h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3|$

$$\Rightarrow \boxed{ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2} = ds_1^2 + ds_2^2 + ds_3^2$$

Arc length along a space curve: $L = \int ds = \int |d\vec{R}|$

Volume element: $dV = ds_1 \, ds_2 \, ds_3 \Rightarrow \boxed{dV = \underbrace{h_1 h_2 h_3}_{|\text{Jacobian}|} du_1 du_2 du_3}$

eg cylindrical coordinates: $dV = \rho \, d\rho \, d\theta \, dz$

spherical coordinates: $dV = r^2 \sin\phi \, dr \, d\phi \, dz$

---

Differential operators in orthogonal curvilinear coordinates:

Gradient: $\nabla f = (\hat{e}_1 \cdot \nabla f)\hat{e}_1 + (\hat{e}_2 \cdot \nabla f)\hat{e}_2 + (\hat{e}_3 \cdot \nabla f)\hat{e}_3$

$$\hat{e}_i \cdot \nabla f = D_{\hat{e}_i} f = \frac{\partial f}{\partial s_i} \quad \leftarrow \text{only } u_i \text{ varies: } ds_i = h_i du_i \quad = \frac{1}{h_i}\frac{\partial f}{\partial u_i}$$

$$\Rightarrow \boxed{\nabla f = \text{grad } f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3}$$

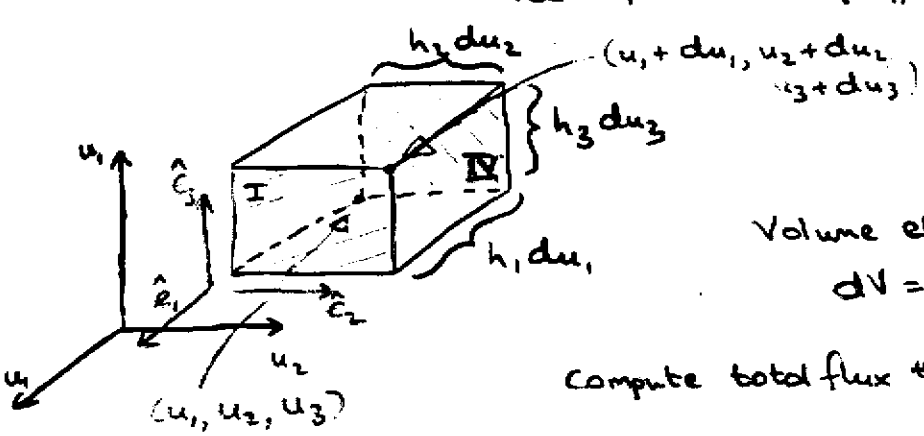
eg spherical coordinates:

$$\boxed{\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\hat{e}_\theta}$$

---

Divergence: — flux per unit volume

Vector field $\vec{F} = F_1(u_1,u_2,u_3)\hat{e}_1 + F_2(u_1,u_2,u_3)\hat{e}_2 + F_3(u_1,u_2,u_3)\hat{e}_3$ (flux density)




Volume element

$dV = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3$

Compute total flux through small box

Contribution to total flux from faces I (normal $\hat{e}_1$) and IV (normal $-\hat{e}_1$)

On I, outward normal $\hat{e}_1$, area $(h_2 du_2)(h_3 du_3) = h_2 h_3 du_2 du_3$

$$\text{Flux} \quad \vec{F}\cdot\hat{n}\, dS\Big|_I = F_1 h_2 h_3 du_2 du_3\Big|_I \leftarrow \text{at } (u_1+du_1, u_2, u_3)$$

On IV, outward normal $-\hat{e}_1$: $\vec{F}\cdot\hat{n}\, dS\Big|_{IV} = -F_1 h_2 h_3 du_2 du_3\Big|_{IV}$ at $(u_1, u_2, u_3)$

Net contribution from I & IV to outward flux:

$$\vec{F}\cdot\hat{n}\, dS\Big|_I + \vec{F}\cdot\hat{n}\, dS\Big|_{IV} = F_1 h_2 h_3 du_2 du_3\Big|_{u_1}^{u_1+du_1} = \frac{\partial}{\partial u_1}(F_1 h_2 h_3)\, du_1 du_2 du_3$$

The contribution to the flux from the other faces is computed similarly. Total net outward flux

$$= \left[\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(F_2 h_3 h_1) + \frac{\partial}{\partial u_3}(F_3 h_1 h_2)\right] du_1 du_2 du_3$$

divide by volume $dV = h_1 h_2 h_3 du_1 du_2 du_3$

$$\Rightarrow \quad \operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(F_2 h_3 h_1) + \frac{\partial}{\partial u_3}(F_3 h_1 h_2)\right]$$

eg in <u>spherical</u> coordinates

$$\boxed{\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, F_\phi) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}}$$

<u>Laplacian</u>: $\nabla\cdot(\nabla f) = \nabla\cdot\left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3\right)$

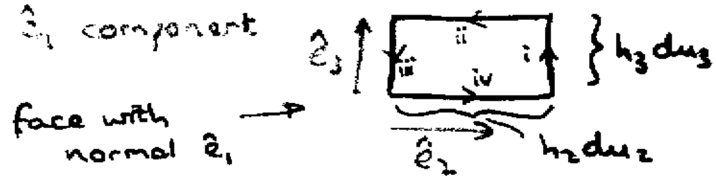
$$\Rightarrow \nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

eg in <u>spherical</u> coordinates

$$\boxed{\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}}$$

## Curl: – circulation per unit area

eg $\hat{e}_1$ component




Total circulation

$$\oint \vec{F}\cdot\vec{T}\,d\ell = \underbrace{F_3}_{\vec{F}\cdot\vec{T}\text{ on } i}\underbrace{h_3\,du_3}_{\text{length of side } i\,=\,ds_3}\Big|_{i\;(u_2+du_2)} - F_3h_3\,du_3\Big|_{iii\;u_2} - F_2h_2\,du_2\Big|_{ii\;u_3+du_3} + F_2h_2\,du_2\Big|_{iv\;u_3}$$

$$= \frac{\partial}{\partial u_2}(h_3F_3)\,du_2\,du_3 - \frac{\partial}{\partial u_3}(h_2F_2)\,du_2\,du_3$$

Area $= h_2h_3\,du_2\,du_3$

$\Rightarrow \hat{e}_1$ component $\quad \text{curl}\,\vec{F}\cdot\hat{e}_1 = \dfrac{1}{h_2h_3}\left[\dfrac{\partial}{\partial u_2}(h_3F_3) - \dfrac{\partial}{\partial u_3}(h_2F_2)\right]$

The $\hat{e}_2$ and $\hat{e}_3$ components of curl $\vec{F}$ are computed similarly

$$\Rightarrow \text{curl}\,\vec{F} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right]\hat{e}_1 + \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_3F_3)\right]\hat{e}_2$$
$$+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right]\hat{e}_3$$

$$\Rightarrow \boxed{\begin{array}{c}\text{curl}\,\vec{F}\\ \shortparallel \\ \nabla\times\vec{F}\end{array} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}}$$

eg in spherical coordinates

$$\boxed{\nabla\times\vec{F} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \hat{e}_r & r\hat{e}_\phi & r\sin\phi\,\hat{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ F_r & rF_\phi & r\sin\phi\,F_\theta \end{vmatrix}}$$

Examples:

- In spherical coordinates

$$f(r,\phi,\theta) = \frac{\cos\phi}{r^2} \Rightarrow \nabla f = -\frac{2\cos\phi}{r^3}\,\hat{e}_r - \frac{\sin\phi}{r^3}\,\hat{e}_\phi + 0\,\hat{e}_\theta$$

- $f(r,\phi,\theta) = \frac{1}{r} = \frac{1}{|\vec{R}|}$ $(r \neq 0)$

$\Rightarrow \nabla f = -\frac{1}{r^2}\hat{e}_r = -\frac{\vec{R}}{|\vec{R}|^3}$ (inverse square force)

$\Rightarrow \nabla^2 f = \nabla\cdot(\nabla f) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$

$\Rightarrow f(r,\phi,\theta) = \frac{1}{r}$ is harmonic $(\nabla^2 f = 0)$ in 3-d, for $r \neq 0$.

---

General orthogonal curvilinear coordinates

- $$\left.\begin{aligned} x &= u_1^2 - u_2^2 \\ y &= 2u_1u_2 \\ z &= u_3 \end{aligned}\right\} \Rightarrow \vec{R} = \underbrace{(u_1^2 - u_2^2)}_{x}\,\hat{\imath} + \underbrace{2u_1u_2}_{y}\,\hat{\jmath} + \underbrace{u_3}_{z}\hat{k}$$

$$\frac{\partial\vec{R}}{\partial u_1} = 2u_1\hat{\imath} + 2u_2\hat{\jmath}, \quad \frac{\partial\vec{R}}{\partial u_2} = -2u_2\hat{\imath} + 2u_1\hat{\jmath}, \quad \frac{\partial\vec{R}}{\partial u_3} = \hat{k}$$

$$\Rightarrow \frac{\partial\vec{R}}{\partial u_1}\cdot\frac{\partial\vec{R}}{\partial u_2} = \frac{\partial\vec{R}}{\partial u_1}\cdot\frac{\partial\vec{R}}{\partial u_3} = \frac{\partial\vec{R}}{\partial u_2}\cdot\frac{\partial\vec{R}}{\partial u_3} = 0$$

$$\frac{\partial\vec{R}}{\partial u_1}\times\frac{\partial\vec{R}}{\partial u_2} = (2u_1\hat{\imath} + 2u_2\hat{\jmath})\times(-2u_2\hat{\imath} + 2u_1\hat{\jmath}) = 4(u_1^2+u_2^2)\hat{k}$$
$$= \underbrace{4(u_1^2+u_2^2)}_{\geq 0}\frac{\partial\vec{R}}{\partial u_3}$$

$\Rightarrow (u_1, u_2, u_3)$ form right-handed orthogonal coordinates.

$$\hat{e}_1 = \frac{u_1\hat{\imath} + u_2\hat{\jmath}}{\sqrt{u_1^2+u_2^2}}, \quad \hat{e}_2 = \frac{-u_2\hat{\imath} + u_1\hat{\jmath}}{\sqrt{u_1^2+u_2^2}}, \quad \hat{e}_3 = \hat{k}$$

$h_i = \left|\frac{\partial\vec{R}}{\partial u_i}\right| \Rightarrow h_1 = h_2 = 2\sqrt{u_1^2+u_2^2}, \quad h_3 = 1$ scale factors

For $f(u_1,u_2,u_3) = u_1u_2 + u_3^2$,

$$\text{grad} f = \nabla f = \frac{1}{2\sqrt{u_1^2+u_2^2}}\,u_2\,\hat{e}_1 + \frac{1}{2\sqrt{u_1^2+u_2^2}}\,u_1\,\hat{e}_2 + 2u_3\,\hat{e}_3$$

# Dyadics

[Return to Cartesian coordinates and a fixed basis $\hat{\imath}, \hat{\jmath}, \hat{k}$ for this section]

For scalar fields $f$, $\nabla^2 f = \nabla\cdot(\nabla f) = \mathrm{div}(\mathrm{grad}\, f)$.

Can we interpret $\nabla^2\vec{F}$ as $\mathrm{div}(\mathrm{grad}\,\vec{F})$ for vector fields $\vec{F}$

ie how do we interpret $\underline{\mathrm{grad}\,\vec{F} = \nabla\vec{F}}$?

(we defined $\nabla^2\vec{F} = (\nabla^2 F_1)\hat{\imath} + (\nabla^2 F_2)\hat{\jmath} + (\nabla^2 F_3)\hat{k}$ )

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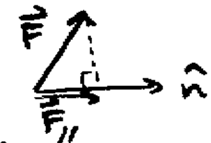
Consider:

$\hat{n}(\hat{n}\cdot\vec{F})$ = projection of vector $\vec{F}$ in the direction of the unit vector $\hat{n}$ $= \vec{F}_{\parallel}$



- define "projection operator in the direction of $\hat{n}$" : $P : \vec{F} \mapsto \hat{n}(\hat{n}\cdot\vec{F})$

maps a vector to a vector

- denote this projection operator by $\underline{\hat{n}\hat{n}}$ :

$$(\hat{n}\hat{n})\cdot\vec{F} \underset{\text{def}}{\equiv} \hat{n}(\hat{n}\cdot\vec{F})$$

Generalize this idea:

Given any two vectors $\vec{A}$, $\vec{B}$, define the $\underline{\text{dyadic}}$ $\vec{A}\vec{B}$ as an $\underline{\text{operator}}$, so that for any vector $\vec{F}$ :

$$\boxed{\begin{aligned} (\vec{A}\vec{B})\cdot\vec{F} &\overset{\text{def}}{\equiv} \vec{A}(\vec{B}\cdot\vec{F}) \\ \text{and}\quad \vec{F}\cdot(\vec{A}\vec{B}) &\equiv (\vec{F}\cdot\vec{A})\vec{B} \end{aligned}}$$

eg dyadic $\hat{\imath}\hat{\imath}$ : projection onto x-axis

$$(\hat{\imath}\hat{\imath})\cdot\vec{F} = \hat{\imath}(\hat{\imath}\cdot\vec{F}) = \hat{\imath}F_1 = F_1\hat{\imath} = (\vec{F}\cdot\hat{\imath})\hat{\imath} = \vec{F}\cdot(\hat{\imath}\hat{\imath})$$

eg $\underline{\text{identity}}$ $\underline{\text{operator}}$ . for any $\vec{F}$

$$(\hat{\imath}\hat{\imath} + \hat{\jmath}\hat{\jmath} + \hat{k}\hat{k})\cdot\vec{F} = \hat{\imath}F_1 + \hat{\jmath}F_2 + \hat{k}F_3 = \vec{F} = \vec{F}\cdot(\hat{\imath}\hat{\imath} + \hat{\jmath}\hat{\jmath} + \hat{k}\hat{k})$$

eg $(\hat{i}\hat{j})\cdot\vec{F} = \hat{i}(\hat{j}\cdot\vec{F}) = \hat{i}F_2 = F_2\hat{i}$ ,
$\vec{F}\cdot(\hat{i}\hat{j}) = (\vec{F}\cdot\hat{i})\hat{j} = F_1\hat{j} \neq F_2\hat{i}$ $\Big\}$ $(\hat{i}\hat{j})\cdot\vec{F} \neq \vec{F}\cdot(\hat{i}\hat{j})$

- in general $\underbrace{(\vec{A}\vec{B})\cdot\vec{F}}_{\text{direction of } \vec{A}} \neq \underbrace{\vec{F}\cdot(\vec{A}\vec{B})}_{\text{direction of } \vec{B}}$ and $\vec{A}\vec{B} \neq \vec{B}\vec{A}$ .

In expanded form

$$\begin{aligned}
\vec{A}\vec{B} &= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) \\
&= A_1B_1\,\hat{i}\hat{i} + A_1B_2\,\hat{i}\hat{j} + A_1B_3\,\hat{i}\hat{k} \\
&\quad + A_2B_1\,\hat{j}\hat{i} + A_2B_2\,\hat{j}\hat{j} + A_2B_3\,\hat{j}\hat{k} \\
&\quad + A_3B_1\,\hat{k}\hat{i} + A_3B_2\,\hat{k}\hat{j} + A_3B_3\,\hat{k}\hat{k}
\end{aligned}$$

Matrix representation

Recall - we can write the components of a vector $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$
[w.r.t. a fixed Cartesian basis $\{\hat{i}, \hat{j}, \hat{k}\}$] as a column vector
(3×1 matrix)

$$\begin{array}{cl} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} & \begin{array}{l} \leftarrow \hat{i} \text{ component} \\ \leftarrow \hat{j} \\ \leftarrow \hat{k} \end{array} \end{array}$$

Similarly, we can write a dyadic $\vec{A}\vec{B}$ as a matrix (w.r.t $\{\hat{i}, \hat{j}, \hat{k}\}$)

$$\begin{array}{cc} & \hat{i}\hat{j} \\ \begin{array}{c} \hat{i}\hat{i} \\ \hat{j}\hat{i} \\ \\ \end{array} & \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix} \end{array} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}$$

- note $(\vec{A}\vec{B})\cdot\vec{F} = \vec{A}(\vec{B}\cdot\vec{F})$

$$\begin{aligned}
&= \hat{i}A_1(B_1F_1 + B_2F_2 + B_3F_3) + \hat{j}A_2(B_1F_1 + B_2F_2 + B_3F_3) \\
&\quad + \hat{k}A_3(B_1F_1 + B_2F_2 + B_3F_3)
\end{aligned}$$

- in matrix form

$$\vec{A}\vec{B} \rightarrow \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix} \underbrace{\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}}_{\vec{F}}$$

eg grad $\vec{F} = \nabla\vec{F}$ can be interpreted as a dyadic:

$$\nabla\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{\imath}F_1 + \hat{\jmath}F_2 + \hat{k}F_3)$$

$$= \frac{\partial F_1}{\partial x}\hat{\imath}\hat{\imath} + \frac{\partial F_2}{\partial x}\hat{\imath}\hat{\jmath} + \frac{\partial F_3}{\partial x}\hat{\imath}\hat{k}$$

$$+ \frac{\partial F_1}{\partial y}\hat{\jmath}\hat{\imath} + \frac{\partial F_2}{\partial y}\hat{\jmath}\hat{\jmath} + \frac{\partial F_3}{\partial y}\hat{\jmath}\hat{k}$$

$$+ \frac{\partial F_1}{\partial z}\hat{k}\hat{\imath} + \frac{\partial F_2}{\partial z}\hat{k}\hat{\jmath} + \frac{\partial F_3}{\partial z}\hat{k}\hat{k}$$

Matrix representation

of $\nabla f$ : $\begin{pmatrix}\partial_x f\\ \partial_y f\\ \partial_z f\end{pmatrix}$ of $\nabla\vec{F}$ : $\begin{pmatrix}\partial_x F_1 & \partial_x F_2 & \partial_x F_3\\ \partial_y F_1 & \partial_y F_2 & \partial_y F_3\\ \partial_z F_1 & \partial_z F_2 & \partial_z F_3\end{pmatrix}$

Dyadic interpretation of div(grad $\vec{F}$) = $\nabla\cdot(\nabla\vec{F})$ :

$$\nabla\cdot(\nabla\vec{F}) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\Big[\hat{\imath}\hat{\imath}\,\partial_x F_1 + \hat{\imath}\hat{\jmath}\,\partial_x F_2 + \hat{\imath}\hat{k}\,\partial_x F_3$$

$$+ \hat{\jmath}\hat{\imath}\,\partial_y F_1 + \hat{\jmath}\hat{\jmath}\,\partial_y F_2 + \hat{\jmath}\hat{k}\,\partial_y F_3$$

$$+ \hat{k}\hat{\imath}\,\partial_z F_1 + \hat{k}\hat{\jmath}\,\partial_z F_2 + \hat{k}\hat{k}\,\partial_z F_3\Big]$$

$$= \hat{\imath}\frac{\partial^2 F_1}{\partial x^2} + \hat{\jmath}\frac{\partial^2 F_2}{\partial x^2} + \hat{k}\frac{\partial^2 F_3}{\partial x^2} + \vec{0}\frac{\partial^2 F_1}{\partial x\partial y} + \cdots$$

$\hat{\imath}\cdot(\hat{\imath}\hat{\jmath}) = (\hat{\imath}\cdot\hat{\imath})\hat{\jmath}$ $\hat{\imath}\cdot(\hat{\jmath}\hat{\imath}) = (\hat{\imath}\cdot\hat{\jmath})\hat{\imath} = 0\hat{\imath} = \vec{0}$

$$+ \hat{\imath}\frac{\partial^2 F_1}{\partial y^2} + \hat{\jmath}\frac{\partial^2 F_2}{\partial y^2} + \hat{k}\frac{\partial^2 F_3}{\partial y^2} + \hat{\imath}\frac{\partial^2 F_1}{\partial z^2} + \hat{\jmath}\frac{\partial^2 F_2}{\partial z^2} + \hat{k}\frac{\partial^2 F_3}{\partial z^2}$$

$$= \hat{\imath}(\nabla^2 F_1) + \hat{\jmath}(\nabla^2 F_2) + \hat{k}(\nabla^2 F_3) = \nabla^2\vec{F}$$

$\Rightarrow$ the dyadic interpretation agrees with our previous expression for $\nabla^2\vec{F}$

## Linear Approximation and Derivative

One dimension: Recall $f(x_0+h) \approx \underbrace{f(x_0) + f'(x_0)h}$ : write $h = \Delta x$ $= x - x_0$

linear approximation to $f$ at $x_0$

Tangent line to $f(x)$ at $x=x_0$: $y = \ell(x) = f(x_0) + f'(x_0)(x-x_0)$

Equivalently: $\underbrace{f(x_0+h) - f(x_0)}_{\text{increment } \Delta f} \approx \underbrace{f'(x_0)h}_{\text{differential } df}$ ← $\Delta x$

We say

$\ell(x) = f(x_0) + f'(x_0)(x-x_0)$ is the best linear approximation to $f(x)$ near $x_0$.

**Def:** $f$ is differentiable at $x_0$ if there exists $L$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L$$

— then we say $L = f'(x_0)$

or equivalently

linear approximation $\ell(x)$

$$\lim_{x \to x_0} \left| \frac{f(x) - \overbrace{f(x_0) - L(x-x_0)}}{x - x_0} \right| = 0$$

— the difference between $f(x)$ and its linear approximation $\ell(x)$ approaches zero as $x \to x_0$ even when divided by $x - x_0$:

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{linear approximation } \ell(x)} + \underbrace{\varepsilon(x,x_0)(x-x_0)}_{\text{error}}$$

where $\varepsilon(x,x_0) \to 0$ as $x \to x_0$

— this expresses the idea that $\ell(x)$ is a "good approximation" to $f(x)$ near $x_0$.

[We write $f(x) = \ell(x) + o(x-x_0)$ ← means $\frac{f(x) - \ell(x)}{x - x_0} \to 0$ as $x \to x_0$]

"little-oh"

Two dimensions: $f: \mathbb{R}^2 \to \mathbb{R}$

Let $z = f(x,y)$ be the graph of $f$, a surface in $\mathbb{R}^3$

ie $g(x,y,z) = f(x,y) - z = 0$ — level surface $g=0$ of $g(x,y,z)$

Normal to the surface: $\nabla g = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} - \hat{k}$

Tangent plane at $\vec{R}_0 = x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}$, where $z_0 = f(x_0,y_0)$:

$$(\vec{R} - \vec{R}_0)\cdot\nabla g = 0 \Rightarrow (x-x_0)\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + (y-y_0)\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = z - z_0$$

tangent plane to graph of $f$ at $(x_0,y_0)$ $\rightarrow$

$$\Rightarrow \quad z = \underbrace{f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}_{\text{Linear approximation } \ell(x,y) \text{ to } f(x,y)}$$

Definition: We say $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0,y_0)$ if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(x_0,y_0)$ and if

$$\frac{f(x,y) - \left\{f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\right\}}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \to 0 \text{ as } (x,y) \to (x_0,y_0)$$

or equivalently

$$f(x,y) = \underbrace{f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}_{\ell(x,y)} + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(x,y) \to (x_0,y_0)$.

This expresses the idea that the tangent plane is a "good approximation" to $f$ near $(x_0,y_0)$.

Notes:
- If $f \in C^1$ ie $f$ has continuous first partial derivatives at $(x_0,y_0)$, $\partial f/\partial x$, $\partial f/\partial y$ are continuous then $f$ is differentiable
- Existence (without continuity) of $\partial f/\partial x$, $\partial f/\partial y$ at $(x_0,y_0)$ does not guarantee differentiability

In vector form: Position vector $\vec{R} = x\hat{\imath} + y\hat{\jmath}$, $\vec{R}_0 = x_0\hat{\imath} + y_0\hat{\jmath}$

$$f(\vec{R}) = f(\vec{R}_0) + \nabla f(\vec{R}_0)\cdot(\vec{R}-\vec{R}_0) + \underbrace{\vec{\varepsilon}}_{\to 0 \text{ as } \vec{R}\to\vec{R}_0}\cdot(\vec{R}-\vec{R}_0)$$

Linear approximation: $(\vec{h} = \vec{R}-\vec{R}_0)$

$$\vec{f}(\vec{R}_0+\vec{h}) \approx f(\vec{R}_0) + \nabla f(\vec{R}_0)\cdot\vec{h}$$

---

Alternative notation: write

$Df(\vec{R}_0) = Df(x_0, y_0)$ as a $1\times 2$ row matrix ← linear operator

$$Df(x_0,y_0) = \left( \frac{\partial f}{\partial x}(x_0,y_0) \quad \frac{\partial f}{\partial y}(x_0,y_0) \right)$$

then

$$f(\vec{R}) = f(\vec{R}_0) + \underbrace{Df(\vec{R}_0)}_{\substack{1\times 2\\ \text{matrix}}}\underbrace{(\vec{R}-\vec{R}_0)}_{2\times 1} + \underbrace{o(|\vec{R}-\vec{R}_0|)}_{\substack{\text{decays to 0 faster}\\ \text{than } |\vec{R}-\vec{R}_0|}}$$

---

For general scalar fields $f: \mathbb{R}^n \to \mathbb{R}$ :

Def: $f$ is <u>differentiable</u> at $\vec{R}_0$ if there exists a constant vector $\vec{c}$ s.t.

$$\lim_{|\vec{h}|\to 0} \frac{f(\vec{R}_0+\vec{h}) - f(\vec{R}_0) - \vec{c}\cdot\vec{h}}{|\vec{h}|} = 0$$

ie $f(\vec{R}_0+\vec{h}) = f(\vec{R}_0) + \vec{c}\cdot\vec{h} + \underbrace{\vec{\varepsilon}}_{\to 0}\cdot\vec{h}$

existence of partial derivatives follows from differentiability → If the partial derivatives of $f$ exist at $\vec{R}_0$, then $\vec{c} = \nabla f(\vec{R}_0)$ $\quad \left[ c_i = \frac{\partial f}{\partial x_i}(\vec{R}_0) \right]$

[that is, $f$ is differentiable if there is a linear function $\vec{c}\cdot\vec{h}$ that approximates the increment $f(\vec{R}_0+\vec{h}) - f(\vec{R}_0)$ so closely that the error is small compared to $|\vec{h}|$ ]

We can write $\vec{c}$ as a row matrix

$$\vec{c}^{\,T} = Df(\vec{R}_0)$$

$\overset{\shortparallel}{[\nabla f(\vec{R}_0)]^T}$ $\quad \Rightarrow \vec{c}\cdot\vec{h} = Df(\vec{R}_0)\,\vec{h}$ . ← derivative : linear operator

For general maps $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ (or $\vec{F}$ : m functions of n variables.)
open set

Def: $F$ is differentiable at $\vec{R}_0 \in U$ if the partial derivatives of each component of $F$ exist at $\vec{R}_0$ and there is a linear operator $T$ ($T : \mathbb{R}^n \to \mathbb{R}^m$) such that

$$\lim_{|\vec{h}| \to 0} \frac{|F(\vec{R}_0 + \vec{h}) - F(\vec{R}_0) - T\vec{h}|}{|\vec{h}|} = 0 \qquad \vec{h} = \vec{R} - \vec{R}_0$$

If we consider $\vec{h}$ as a $n \times 1$ matrix (column vector) then we can represent $T$ as a $m \times n$ matrix (linear operator).

We write $T = DF(\vec{R}_0)$ with matrix elements $T_{ij} = \frac{\partial F_i}{\partial x_j}(\vec{R}_0)$

"derivative of $F$ at $\vec{R}_0$"

$$\Rightarrow F(\vec{R}_0 + \vec{h}) = F(\vec{R}_0) + DF(\vec{R}_0)\vec{h} + o(|\vec{h}|)$$

$$DF(\vec{R}_0) = \left.\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}\right|_{\vec{R}_0}$$

Matrix of partial derivatives of $F$ at $\vec{R}_0$ (Jacobian matrix)

(Note: if $m=1$, $DF(\vec{R}_0)$ is a row matrix; the corresponding (column) vector is $\nabla F(\vec{R}_0)$, with $DF(\vec{R}_0)\vec{h} = \nabla F(\vec{R}_0) \cdot \vec{h}$ )

eg for a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$D\vec{F}(\vec{R}_0)$ is represented by the matrix

$$\underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix}}_{\text{Note } \operatorname{div} \vec{F} = \operatorname{trace}(D\vec{F})}$$

eg $\vec{F}$ could represent a coordinate transformation $(u,v,w) = \vec{F}(x,y,z)$

Jacobian matrix $\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}$

Note on notational conventions:

Observe that the conventions for dyadics and the derivative matrix (operator) defined here are different:

$$T_{ij} = (D\vec{F})_{ij} = \frac{\partial F_i}{\partial x_j}, \quad (\nabla \vec{F})_{ij} = \frac{\partial F_j}{\partial x_i} \quad \Rightarrow \quad D\vec{F}(\vec{R}_0) = [\nabla \vec{F}(\vec{R}_0)]^T$$

The derivative $D\vec{F}(\vec{R}_0)$ is an operator (linear transformation):

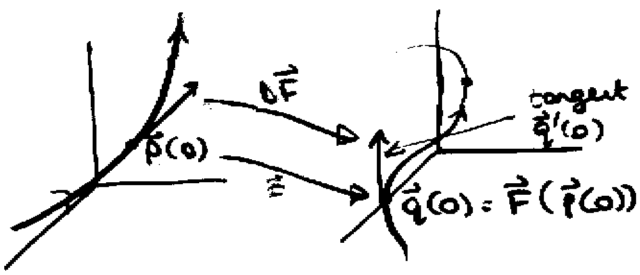
maps vectors to vectors.

Geometric interpretation:

eg Consider a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ maps a region $U \subset \mathbb{R}^3$ to another region

and a curve $\vec{R} = \vec{p}(t) \quad : \mathbb{R} \to \mathbb{R}^3$

Let $\vec{q}(t) = \vec{F}(\vec{p}(t))$ : image of path under the map $\vec{F}$




Components of tangent: (chain rule!)

$$q_i'(t) = \frac{\partial F_i}{\partial x}(\vec{p}(t))\frac{dp_1}{dt} + \frac{\partial F_i}{\partial y}\frac{dp_2}{dt} + \frac{\partial F_i}{\partial z}\frac{dp_3}{dt}$$

$$= \vec{p}'(t) \cdot \nabla F_i(\vec{p}(t))$$

$$\Rightarrow \quad \vec{q}'(t) = \vec{p}'(t) \cdot \nabla \vec{F}(\vec{p}(t)) \quad ;$$

equivalently, $\vec{q}'(t) = D\vec{F}(\vec{p}(t))\,\vec{p}'(t)$ ← expression of chain rule

matrix multiplication $\qquad \dfrac{d}{dt}\vec{F}(\vec{p}(t)) = D\vec{F}(\vec{p}(t))\dfrac{d\vec{p}}{dt}$

Observe: Map $\vec{F}$ (vector field) : maps points $\vec{R} \in \mathbb{R}^3$ to points (position vector to position vector)

Derivative matrix $D\vec{F}$ : maps tangent vectors of a path $\vec{p}(t)$ to tangent vectors of the corresponding image path $\vec{q}(t)$ (velocity vector to velocity vector)

— — — — —

A function $g(\vec{R}_0; \vec{h})$ is a $k^{th}$ order approximation of $f$ at the point $\vec{R}_0$ if $\qquad k$: positive integer

$$\lim_{\vec{h} \to \vec{0}} \frac{|f(\vec{R}_0 + \vec{h}) - g(\vec{R}_0; \vec{h})|}{|\vec{h}|^k} = 0$$

eg linear approximation: $f(\vec{R}_0 + \vec{h}) = \underbrace{f(\vec{R}_0) + Df(\vec{R}_0)\vec{h}}_{g(\vec{R}_0;\vec{h})} + o(|\vec{h}|)$

## Taylor Polynomials and Quadratic Approximation

Higher-order best approximations to $f$ may be obtained using Taylor polynomials.

Recall: in one variable, for $f \in C^n$, $f = f(x)$ ← $n$ continuous derivatives

$h = x - x_0$

the Taylor polynomial is given by

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$

$$\dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$p_n(x)$ provides the best approximation to $f$ among all $n^{th}$ order polynomials. Error: $R_n(x) = f(x) - p_n(x)$ (various forms of the remainder are known)

↖ remainder

eg If $f^{(n+1)}(x_0)$ is continuous, $\lim_{x \to x_0} \left| \frac{f(x) - p_n(x)}{(x-x_0)^n} \right| = 0$ $n^{th}$ order approximation

(note: if $f \in C^\infty$ and the series converges as $n \to \infty$ to $f(x)$ for $|x - x_0| < \varepsilon$, some $\varepsilon > 0$, then $f$ is <u>analytic</u> at $x_0$)

---

We may obtain Taylor polynomials in higher dimensions from the 1-d formulas:

$\vec{R}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$
$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

Consider a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$

Let $\vec{R} - \vec{R}_0 = s\hat{u}$

fixed ↗; ↖ fixed unit vector; ↑ distance

Define $g(s) = f(\vec{R}_0 + s\hat{u})$ ← a function of one variable: expand $f(\vec{R})$ about $\vec{R}_0$ $\Leftrightarrow$ expand $g(s)$ about $s=0$

$= f(\vec{R})$

Then $g(s) = g(0) + g'(0)s + \frac{1}{2}g''(0)s^2 + \dots$

where $g'(0) = \frac{d}{ds} f(\vec{R}_0 + s\hat{u})\Big|_{s=0}$ rate of change of $f$ w.r.t. distance in direction $\hat{u}$

$=$ directional derivative $D_{\hat{u}} f(\vec{R}_0) = \hat{u} \cdot \nabla f(\vec{R}_0)$

$$g'(0) = \hat{u}\cdot\nabla f(\vec{R}_0)$$

$$g''(0) = \hat{u}\cdot\nabla[\hat{u}\cdot\nabla f(\vec{R}_0)] = \hat{u}\cdot\nabla[\nabla f(\vec{R}_0)\cdot\hat{u}]$$

$$= \hat{u}\cdot\underbrace{\nabla\nabla f(\vec{R}_0)}_{\text{dyadic} \leftarrow \text{the Hessian of } f \text{ at } \vec{R}_0}\cdot\hat{u}$$

Hessian

$$\nabla\nabla f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f$$

$$\left.\begin{aligned} &= \hat{\imath}\hat{\imath}\frac{\partial^2 f}{\partial x^2} + \hat{\imath}\hat{\jmath}\frac{\partial^2 f}{\partial x\partial y} + \hat{\imath}\hat{k}\frac{\partial^2 f}{\partial x\partial z} \\ &+ \hat{\jmath}\hat{\imath}\frac{\partial^2 f}{\partial y\partial x} + \hat{\jmath}\hat{\jmath}\frac{\partial^2 f}{\partial y^2} + \hat{\jmath}\hat{k}\frac{\partial^2 f}{\partial y\partial z} \\ &+ \hat{k}\hat{\imath}\frac{\partial^2 f}{\partial z\partial x} + \hat{k}\hat{\jmath}\frac{\partial^2 f}{\partial z\partial y} + \hat{k}\hat{k}\frac{\partial^2 f}{\partial z^2} \end{aligned}\right\} \text{Hessian of } f$$

Substituting $g'(0)$, $g''(0)$ into the expansion for $g(s)$, and using $s\hat{u} = \vec{R} - \vec{R}_0$, we find the second-order Taylor polynomial for $f$ around $\vec{R}_0$:

$$P_2(\vec{R}) = \underbrace{f(\vec{R}_0)}_{g(0)} + \underbrace{\underbrace{(\vec{R}-\vec{R}_0)}_{s\hat{u}}\cdot\nabla f(\vec{R}_0)}_{s g'(0)} + \tfrac{1}{2}\underbrace{(\vec{R}-\vec{R}_0)}_{s\hat{u}}\cdot\nabla\nabla f(\vec{R}_0)\cdot(\vec{R}-\vec{R}_0)$$

In matrix notation (subscripts denote differentiation eg $f_x = \frac{\partial f}{\partial x}$, $f_{xy} = \frac{\partial^2 f}{\partial y\partial x}$ )

$$\begin{aligned} P_2(x,y,z) &= f(x_0,y_0,z_0) \\ &+ (x-x_0,\ y-y_0,\ z-z_0)\left.\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}\right|_{(x_0,y_0,z_0)} \\ &+ \tfrac{1}{2}(x-x_0,\ y-y_0,\ z-z_0)\underbrace{\left.\begin{pmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}\right|_{(x_0,y_0,z_0)}}_{\text{Hessian matrix}}\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} \end{aligned}$$

(if $f \in C^2$, the equality of mixed partials $\frac{\partial^2 f}{\partial x_i\partial x_j} = \frac{\partial^2 f}{\partial x_j\partial x_i}$ implies that the Hessian is symmetric.)

Note: • we can also write the Hessian as $D^2 f(\vec{R}_0)$ — this is the analogue of the derivative $Df(\vec{R}_0)$. ← matrix entries $[D^2 f(\vec{R}_0)]_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{R}_0)$

• the Laplacian is the trace of the Hessian matrix: $\nabla^2 f = \mathrm{tr}(D^2 f) = \mathrm{tr}(\nabla\nabla f)$

Second-order (quadratic) approximation

$$f(\vec{R}_0+\vec{h}) = f(\vec{R}_0) + \underbrace{\vec{h}\cdot\nabla f(\vec{R}_0)}_{Df(\vec{R}_0)\vec{h}} + \underbrace{\tfrac{1}{2}\vec{h}\cdot\nabla\nabla f(\vec{R}_0)\cdot\vec{h}}_{\frac{1}{2}\vec{h}\cdot D^2 f(\vec{R}_0)\vec{h}} + R_2(\vec{R}_0,\vec{h})$$

↑ remainder/error

where $\lim_{\vec{h}\to\vec{0}} \frac{|R_2(\vec{R}_0,\vec{h})|}{|\vec{h}|^2} = 0$

(in terms of components)

$$\Rightarrow f(\vec{R}_0+\vec{h}) = f(\vec{R}_0) + \sum_{i=1}^{3} h_i \frac{\partial f}{\partial x_i}(\vec{R}_0) + \frac{1}{2!}\sum_{i,j=1}^{3} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{R}_0) + R_2(\vec{R}_0,\vec{h})$$

$(\vec{h} = \vec{R} - \vec{R}_0)$

– this formula is readily generalized to scalar fields on $\mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$, and to higher-order Taylor approximations.

In general, the Taylor expansion in $\mathbb{R}^3$ to $\ell^{th}$ order about $(0,0,0)$ is $f(x,y,z) = P_\ell(x,y,z) + R_\ell(x,y,z)$,

where

$$P_\ell(x,y,z) = \sum_{k=0}^{\ell} \frac{1}{k!}\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\}^k f(0,0,0)$$

f and all its derivatives evaluated at $(0,0,0)$ ⟶

$$= f + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2} + \frac{1}{2}z^2\frac{\partial^2 f}{\partial z^2} + xy\frac{\partial^2 f}{\partial x\partial y} + xz\frac{\partial^2 f}{\partial x\partial z} + yz\frac{\partial^2 f}{\partial y\partial z} + \cdots$$

Taylor polynomial about $(x_0, y_0, z_0)$:

$$P_\ell(x,y,z) = \sum_{k=0}^{\ell} \frac{1}{k!}\left\{(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y} + (z-z_0)\frac{\partial}{\partial z}\right\}^k f(x_0,y_0,z_0)$$

eg $f(x,y,z) = -x^4 - 2y^2 - 4z^4 + 2z^2$ :

gradient $\nabla f = -4x^3\,\hat{\imath} - 4y\,\hat{\jmath} + (4z - 16z^3)\,\hat{k} = \begin{pmatrix} -4x^3 \\ -4y \\ 4z-16z^3 \end{pmatrix}$

representation w.r.t. basis $\{\hat{\imath},\hat{\jmath},\hat{k}\}$

Hessian $\nabla\nabla f = \begin{pmatrix} -12x^2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4-48z^2 \end{pmatrix}$

At $(0,0,0)$, $f = 0$, $\nabla f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$, $\nabla\nabla f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\Rightarrow$ Second-order Taylor polynomial $P_2(x,y,z) = \frac{1}{2}(\vec{R}-\vec{0}).\nabla\nabla f.(\vec{R}-\vec{0})$
$= -2y^2 + 2z^2$

At $(0,0,1/2)$ ie $\vec{R}_0 = \frac{1}{2}\hat{k}$, $f = \frac{1}{4}$, $\nabla f = \vec{0}$, $\nabla\nabla f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{pmatrix}$

$$\Rightarrow P_2(x,y,z) = \frac{1}{4} + \frac{1}{2}(x, y, z-\tfrac{1}{2})\begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z-\frac{1}{2} \end{pmatrix}$$
$$= \frac{1}{4} - 2y^2 - 4(z-\tfrac{1}{2})^2$$

---

Def: • $\vec{R}_0$ is a <u>critical point</u> of $f$ if grad $f = 0$ at $\vec{R}_0$ : $\nabla f(\vec{R}_0) = \vec{0}$

- note .. both $(0,0,0)$ and $(0,0,1/2)$ in the above example are critical points
  - $(0,0,0)$ is neither a local maximum nor a minimum since $f$ decreases in the $y$ direction, increases in the $z$ direction away from $(0,0,0)$ : <u>saddle point</u>
  - $(0,0,1/2)$ is a local maximum

- $\vec{R}_0$ is a local maximum of $f$ if it is a critical point of $f$ and $f(\vec{R}_0) > f(\vec{R})$ for all $\vec{R}$ in some neighbourhood of $\vec{R}_0$

$\vec{h} = \vec{R} - \vec{R}_0$

ie $f(\vec{R}) = f(\vec{R}_0) + \underbrace{\nabla f(\vec{R}_0)}_{\vec{0}} \cdot \vec{h} + \frac{1}{2}\vec{h}\cdot \nabla\nabla f(\vec{R}_0)\cdot \vec{h}$ + higher order terms

$\Rightarrow$ the Hessian at $\vec{R}_0$ describes the behaviour for small deviations from $\vec{R}_0$

At a maximum, we need $\vec{h}\cdot\nabla\nabla f(\vec{R}_0)\cdot\vec{h} \le 0$ for all $\vec{h}$

(the Hessian matrix is negative semi-definite)

If the Hessian matrix is negative definite, i.e. $\vec{h}\cdot\nabla\nabla f(\vec{R}_0)\cdot\vec{h} < 0$ for all $\vec{h} \neq \vec{0}$,

then $\vec{R}_0$ is a strict local maximum

- $\vec{R}_0$ is a local minimum of $f$ if $\nabla f(\vec{R}_0) = \vec{0}$, and $f(\vec{R}_0) < f(\vec{R})$ for all nearby $\vec{R}$

Necessary condition: $\nabla\nabla f(\vec{R}_0)$ is positive semi-definite
$\vec{h}\cdot\nabla\nabla f(\vec{R}_0)\cdot\vec{h} \ge 0$

Sufficient condition: Hessian is positive definite : $\vec{h}\cdot\nabla\nabla f(\vec{R}_0)\cdot\vec{h} > 0$

– generalized second derivative test

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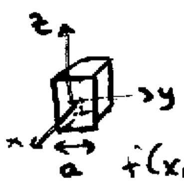
eg Use the second-order Taylor polynomial to estimate the difference between the value of $f(x,y,z)$ at $\vec{0}$ and its average value throughout the interior of a cube, side $a$, centred at $\vec{0}$.




$$f(x,y,z) = f(0,0,0) + xf_x + yf_y + zf_z + \tfrac{1}{2}x^2 f_{xx} + \tfrac{1}{2}y^2 f_{yy} + \tfrac{1}{2}z^2 f_{zz}$$
$$+ xy f_{xy} + xz f_{xz} + yz f_{yz} + \text{h.o.t.}$$

derivatives at $(0,0,0)$ — higher order terms

Average $\bar{f} = f_{av} = \frac{1}{a^3}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} f(x,y,z)\,dx\,dy\,dz$

Average of $f(0,0,0)$ is $f(0,0,0)$. $\int_{-a/2}^{a/2} x\,dx = 0 \Rightarrow$ average of $xf_x$ is $0$

(similarly, average of $yf_y$, $zf_z$, $xyf_{xy}$, $xzf_{xz}$, $yzf_{yz} = 0$)

$\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} x^2\,dx = a^2\cdot\frac{1}{3}x^3\Big|_{-a/2}^{a/2} = \frac{a^5}{12} \Rightarrow$ average of $x^2, y^2, z^2$ is $\frac{a^2}{12}$ $\Rightarrow f_{av} \approx f(0,0,0) + \frac{a^2}{24}(f_{xx} + f_{yy} + f_{zz})$

$\nabla^2 f$

– Laplacian measures difference between $f$ and $f_{av}$