Line Integrals

- integration along a space curve

C : a smooth space curve, parametrized as $C : \vec{R} = \vec{R}(t)$
$a \le t \le b$

C has endpoints $P : \vec{R} = \vec{R}(a)$, $Q : \vec{R} = \vec{R}(b)$

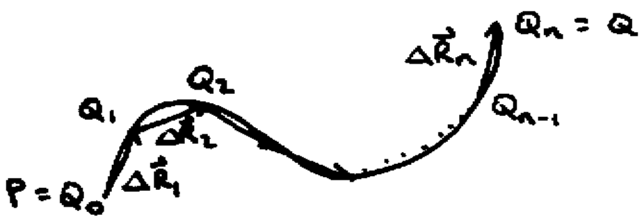
In Cartesian coordinates $\vec{R}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$

$x(t), y(t), z(t) \in C^1$ - or piecewise smooth

Integration of a scalar field f along C :

eg f represents density (mass per unit length) along C
- compute total mass of C

- in Cartesian coordinates $f(x,y,z)$ - assume f is (piecewise) continuous ie $f(\vec{R}(t))$ (piecewise) continuous on $a \le t \le b$



Partition $[a,b]$ into n subintervals

$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$

Q_j has position vector $\vec{R}(t_j)$, $0 \le j \le n$

Length of j^{th} segment $1 \le j \le n$

$$\Delta s_j = |\Delta\vec{R}_j| = |\vec{R}_j - \vec{R}_{j-1}| = |\Delta x_j\,\hat{\imath} + \Delta y_j\,\hat{\jmath} + \Delta z_j\,\hat{k}|$$

$$\underset{\text{mean value theorem}}{=} |\vec{R}'(\tau_j)|\,\Delta t_j \quad \text{for some } \tau_j \in [t_{j-1}, t_j]$$

$$\Delta t_j = t_j - t_{j-1}$$

(recall speed $v(t) = |\vec{R}'(t)| = \frac{ds}{dt}$

element of arc length $ds = |d\vec{R}| = |\vec{R}'(t)|\,dt = v(t)dt$
$= [x'(t)^2 + y'(t)^2 + z'(t)^2]^{1/2}dt$)

Mass of j^{th} segment $f(\vec{R}(\tau_j'))\,\Delta s_j$ some $\tau_j' \in [t_{j-1}, t_j]$

(f continuous, Δt_j small $\Rightarrow f(\vec{R}(t))$ approx. constant on $t_{j-1} \le t \le t_j$)

Total mass

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\vec{R}(\tau_j'))\Delta s_j = \lim_{n\to\infty} \sum_{j=1}^{n} f(\vec{R}(\tau_j'))\,|\vec{R}'(\tau_j)|\Delta t_j$$

integral - a limit of Riemann sums
($n\to\infty$, $\max_j \Delta t_j \to 0$)

$$= \int_a^b f(\vec{R}(t))\,|\vec{R}'(t)|\,dt$$

Def.

The path integral (line integral: integral of f along the smooth space curve $C: \vec{R} = \vec{R}(t)$, $a \le t \le b$) ← continuous is

$$\int_C f\, ds = \int_a^b f(\vec{R}(t))\, |\vec{R}'(t)|\, dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

If C is piecewise smooth (C') and/or $f(\vec{R}(t))$ is piecewise continuous, ← at most a finite no. of jump discontinuities
subdivide the interval $[a,b]$ into pieces, on each of which $f(\vec{R}(t))\,|\vec{R}'(t)|$ is continuous; the line integral is defined as the sum of the integrals along the pieces.

– assume below all functions are (piecewise) continuous, all paths (piecewise) smooth

eg if $f \equiv 1$, we obtain the length of the curve C:

$$L(C) = L = \int_C ds = \int_C |d\vec{R}| = \int_a^b |\vec{R}'(t)|\, dt$$

Average value of f along C: $\displaystyle f_{ave} = \frac{\int_C f\, ds}{L(C)} = \frac{\int_C f\, ds}{\int_C ds}$

Line (path) integrals with respect to coordinate variables

$$\int_C f(x,y,z)\, dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(\vec{R}(\tau_j'))\, \Delta x_j = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$$

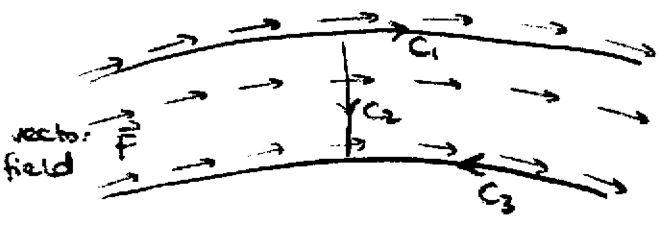
– similarly, integrals w.r.t. y, z

If F_1, F_2, F_3 are continuous functions of x, y, z, we define

$$\int_C \underbrace{F_1\, dx + F_2\, dy + F_3\, dz}_{\text{a "differential form"}} = \int_C F_1\, dx + \int_C F_2\, dy + \int_C F_3\, dz$$

Integration of a vector field $\vec{F}$ along C

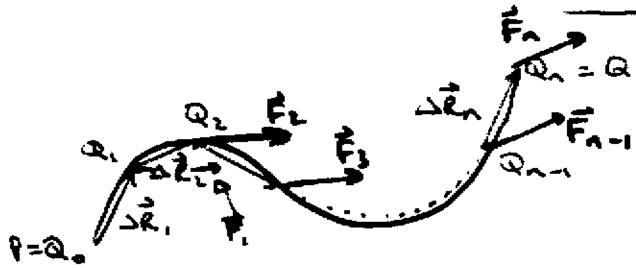
the line integral is a measure of the degree to which the curve C "lines up" with the vector field $\vec{F}$.



Along C_1, line integral $\int_{C_1} \vec{F}\cdot d\vec{R} > 0$

Along C_2, $\int_{C_2} \vec{F}\cdot d\vec{R} = 0$

Along C_3, $\int_{C_3} \vec{F}\cdot d\vec{R} < 0$



As before, partition the curve $C: \vec{R} = \vec{R}(t)$

$$\vec{F}_j = \vec{F}(Q_j) = \vec{F}(\vec{R}(t_j))$$

$$\vec{F}\cdot\Delta\vec{R} = \underbrace{|\vec{F}|}_{\text{magnitude of } \vec{F}} \underbrace{\left[\left(\frac{\vec{F}}{|\vec{F}|}\right)\cdot\Delta\vec{R}\right]}_{\text{component of displacement } \Delta\vec{R} \text{ in direction of } \vec{F}} = \underbrace{\left[\vec{F}\cdot\frac{\Delta\vec{R}}{\Delta s}\right]}_{\text{component of } \vec{F} \text{ in direction of curve } \Delta\vec{R}\ F_{\parallel}} \underbrace{\Delta s}_{\text{element of distance } \Delta s = |\Delta\vec{R}|}$$

At Q_j, $\vec{F}_j\cdot\Delta\vec{R}_j$ measures alignment of field with curve

Define: The <u>line integral</u> of $\vec{F}$ along the curve $C: \vec{R} = \vec{R}(t)$ is

$$\int_C \vec{F}\cdot d\vec{R} = \lim_{\substack{n\to\infty \\ \max|\Delta\vec{R}_j|\to 0}} \sum_{j=1}^{n} \vec{F}_j\cdot\Delta\vec{R}_j$$

If $\vec{F}$ is (piecewise) continuous, the limit exists and is independent of the partition of $[a,b]$.

Application: <u>Force</u> field $\vec{F}$:

Work done by force $\vec{F}$ acting through displacement $\Delta\vec{R}$ is $\Delta W = \vec{F}\cdot\Delta\vec{R}$
$\Rightarrow$ work done on a particle moving along the curve C is

$$W = \int_C \vec{F}\cdot d\vec{R}$$

Notation: • sometimes write $\int_C \vec{F}\cdot d\vec{l}$ or $\int_C \vec{F}\cdot d\vec{s}$ or $\int_C \vec{F}\cdot d\vec{r}$

• Unit tangent vector $\vec{T} = \dfrac{d\vec{R}}{ds} \Rightarrow d\vec{R} = \vec{T}\,ds$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{R} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C F_{\parallel}\,ds$$

$F_{\parallel} = F_t = \vec{F}\cdot\vec{T}$: tangential component of $\vec{F}$

$\Rightarrow$ line integral of $\vec{F}$ is path integral of tangential component of $\vec{F}$ along C.

$\Rightarrow$ Work $= \int_C$ (tangential force component) ds

• In Cartesian coordinates $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$
$d\vec{R} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{R} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

line integral of differential form

• If C is closed, $P = Q$ (ie $\vec{R}(a) = \vec{R}(b)$)
often write $\oint_C \vec{F}\cdot d\vec{R}$: circulation of $\vec{F}$ about C

Compute line integrals using a parametrization $\vec{R}(t)$, $a \le t \le b$ of the curve C :

Parametrization: $\vec{R} = \vec{R}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$, $a \le t \le b$

$$\Rightarrow d\vec{R} = \underbrace{\vec{R}'(t)}_{\vec{v}(t)}\,dt = \underbrace{\frac{\vec{R}'(t)}{|\vec{R}'(t)|}}_{\vec{T}(t)}\,\underbrace{|\vec{R}'(t)|\,dt}_{ds}$$

Definition of line integral:

$$\Rightarrow \int_C \vec{F}\cdot d\vec{R} = \int_a^b \vec{F}(\vec{R}(t))\cdot\vec{R}'(t)\,dt$$

← an ordinary definite integral

$$= \int_a^b \left[\vec{F}(\vec{R}(t))\cdot\vec{T}(t)\right]|\vec{R}'(t)|\,dt$$

in terms of arc length parametrization → $= \int_0^L \vec{F}(\vec{R}(s))\cdot\vec{T}(s)\,ds$

Note: $\int_C \vec{F}\cdot d\vec{R}$ is independent of choice of parametrization of C, provided orientation is preserved.

In terms of components

$$\int_C \vec{F}\cdot d\vec{R} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int_a^b \Big[F_1(x(t),y(t),z(t))\frac{dx}{dt} + F_2(x(t),y(t),z(t))\frac{dy}{dt}$$

$$+ F_3(x(t),y(t),z(t))\frac{dz}{dt}\Big]\,dt$$

Examples:

- Path integral of $f(x,y,z) = x^2+y^2+z^2$ along the helix
  $C:\ \vec{R}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}\ , \quad 0\le t\le 2\pi$

  $\vec{R}'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}\ ,\ |\vec{R}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

  $f(\vec{R}(t)) = \cos^2 t + \sin^2 t + t^2 = t^2+1$ $\qquad \Rightarrow ds = \sqrt{2}\,dt$

  $$\Rightarrow \int_C f\,ds = \int_0^{2\pi} (t^2+1)\sqrt{2}\,dt = \sqrt{2}\left(\tfrac{1}{3}t^3+t\right)\Big|_0^{2\pi} = \frac{2\sqrt{2}\pi}{3}(4\pi^2+3)$$

- Force field $\vec{F} = x^2\,\hat{\imath} + y\,\hat{\jmath} + (xz-y)\,\hat{k}$

  Compute work done in moving a particle from $P:(0,0,0)$ to $Q(1,2,4)$

  – along the line segment C_1 joining P and Q:

  Direction $\vec{v} = \vec{PQ} = \vec{R}_Q - \vec{R}_P = \hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

  $\Rightarrow$ line is $\vec{R}(t) = \vec{R}_P + t\vec{v} = \vec{0} + t(\hat{\imath}+2\hat{\jmath}+4\hat{k})\ ,\ 0\le t\le 1$

  $\Rightarrow x(t) = t,\quad y(t) = 2t,\quad z(t) = 4t$

  $$\text{Work} = \int_{C_1}\vec{F}\cdot d\vec{R} = \int_{C_1} x^2dx + y\,dy + (xz-y)\,dz$$

  $$= \int_0^1 \left[x(t)^2x' + y(t)\,y' + (x(t)z(t)-y(t))\,z'\right]dt$$

  $$= \int_0^1 \left[t^2\cdot 1 + 2t\cdot 2 + (t\cdot 4t - 2t)\cdot 4\right]dt$$

  $$= \int_0^1 (17t^2-4t)\,dt = \left(\tfrac{17}{3}t^3 - 2t^2\right)\Big|_0^1 = \frac{11}{3}$$

  – along the curve C_2: $x(t) = t^2,\ y(t) = 2t,\ z(t) = 4t^3 \quad 0\le t\le 1$

  $$\text{Work} = \int_{C_2}\vec{F}\cdot d\vec{R} = \int_0^1\left[t^4\cdot 2t + 2t\cdot 2 + (4t^5-2t)12t^2\right]dt = \frac{7}{3}$$

$\Rightarrow$ in general, line integral depends on path

Domains ; Simply Connected Domains

Definitions (topology)

1. Given a point P (position vector $\vec{R}_P$), and $\varepsilon > 0$, an ε-neighbourhood of P is the set of points Q whose distance from P is less than ε

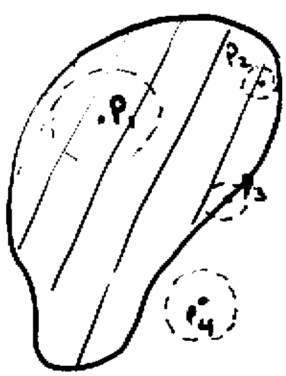
ie $N_\varepsilon(P) = \{ Q : |\vec{R}_Q - \vec{R}_P| < \varepsilon \}$

[eg in $\mathbb{R}^2$, $N_\varepsilon(P)$ contains a disk ie points inside (not on) the circle, centre P, radius ε

in $\mathbb{R}^3$, $N_\varepsilon(P)$ is the interior of a sphere, radius ε]

2. Given a region R, P is an interior point of R if $\exists\ \varepsilon > 0$ s.t. the ε-nhd of P lies completely in R

ie $N_\varepsilon(P) \subset R$ for some $\varepsilon > 0$



P_1, P_2 : interior points
P_3 : boundary point
P_4 : exterior point

P is a boundary point of R if every ε-nhd of P contains both points in R and points outside R

ie $\forall\ \varepsilon > 0$, $N_\varepsilon(P)$ contains at least one point in R, at least one point not in R

P is an exterior point of R if it is neither an interior point nor a boundary point

ie $\exists\ \varepsilon > 0$ s.t. $N_\varepsilon(P) \cap R = \phi$ ← intersection is empty

– ε-nhd of P is disjoint from R

The set of boundary points of R is called the boundary of R – denoted ∂R

3. A region R is <u>open</u> if every point of R is an interior point
ie R does not contain its boundary, $R \cap \partial R = \phi$

[$\Rightarrow$ if a vector field $\vec{F}$ is defined on an open region R, then if $\vec{F}$ is defined at P, it is also defined in an ε-nhd of P for some $\varepsilon > 0$]

The set of all interior points and boundary points of a region R is called the <u>closure</u> of R - denoted $\bar{R}$ $\quad \bar{R} = R \cup \partial R$

A region R is <u>closed</u> if it equals its closure, $R = \bar{R}$
ie a closed region R contains its boundary
Equivalently, R is closed if the <u>complement</u> of R is open
the set of points not in R

[We shall consider only open regions]

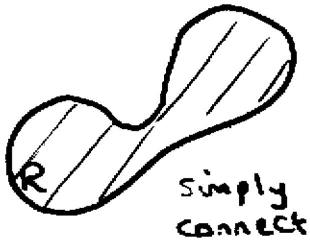
4. An open region R is <u>connected</u> if, given any two points P, Q in R, there is a smooth curve C lying entirely in R that joins P to Q

not connected

A <u>domain</u> : a region R that is both open and connected

[The region of definition of scalar and vector fields will always be a domain.]

5. A domain R is <u>simply connected</u> if every closed curve lying in R can be continuously shrunk to a point in R without the curve ever leaving R.
Equivalently, if C_1, C_2 are any two curves connecting any two points P, Q in R, then C_1 can be continuously deformed to C_2 without passing through points outside the domain R
ie R has "no holes"



simply connected

multiply connected
(not simply connected)

C_1 cannot be shrunk to a point

C_2 cannot be deformed to C_3 within R

When a closed curve C is shrunk to a point, it generates a surface which has the original curve C as its boundary

– thus: a domain R is simply connected if for any closed curve C in R, there is a surface S contained entirely in R with boundary $\partial S = C$.

[much work is required to make this mathematically precise]

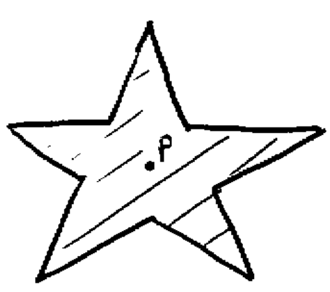
Examples in $\mathbb{R}^3$: • a torus ("doughnut") is not simply connected (has a hole)

• the region inside a cylinder is simply connected (eg $x^2+y^2 < \rho^2$)

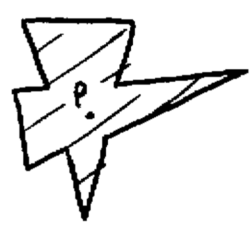
the region outside a cylinder is connected, but not simply connected (eg $x^2+y^2 > \rho^2$)

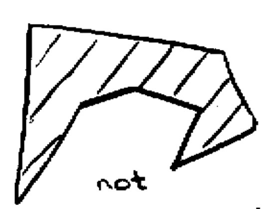
• the region between two concentric spheres is simply connected (eg $a<r<b \Leftrightarrow a^2 < x^2+y^2+z^2 < b^2$)

6. A domain R is star-shaped if there is a point P in R so that for any point Q in R, the line segment PQ lies entirely within R



star-shaped





not star-shaped

Conservative Fields

$\vec{F}$ - a continuous vector field in a domain D (open, connected region)

$$\vec{F} = F_1 \hat{\imath} + F_2 \hat{\jmath} + F_3 \hat{k} \qquad (F_1, F_2, F_3 \text{ - continuous scalar fields in } D)$$

$\vec{F}$ is continuously differentiable in D (written $\vec{F} \in C^1(D)$)

if $D\vec{F}$ is continuous in D $\Leftrightarrow$ $\dfrac{\partial F_i}{\partial x_j}$ is continuous in D $(i,j = 1,2,3)$

If $\vec{F} \in C^1(D)$, then $\left.\begin{array}{l} \text{div } \vec{F} \text{ is a continuous scalar field} \\ \text{curl } \vec{F} \text{ is a continuous vector field} \end{array}\right\}$ in D.

In general, the line integral $\int_C \vec{F} \cdot d\vec{R}$ depends on the endpoints P, Q of C, and on the path.

Def: If $\int_C \vec{F} \cdot d\vec{R}$ is the same for every path C (in D) connecting P, Q, we say the line integral is <u>independent of the path</u> in D (depends only on endpoints); in this case we may write $\int_P^Q \vec{F} \cdot d\vec{R}$

$\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C_2} \vec{F} \cdot d\vec{R}$

Def: A vector field $\vec{F}$ is <u>conservative</u> in a domain D if there exists a scalar field ϕ, defined in D, so that $\vec{F} = \text{grad } \phi$

$\boxed{\vec{F} = \nabla \phi}$ $\Leftrightarrow$ $\vec{F}$ conservative

ϕ is a <u>potential function</u> for $\vec{F}$

Note: the potential function ϕ for $\vec{F}$, if it exists, is unique only up to an additive constant : $\phi + c$ is also a potential for any $c \in \mathbb{R}$
$\Rightarrow$ we can choose the zero of potential arbitrarily

Def: A vector field $\vec{F}$ is <u>irrotational</u> in a domain D if $\text{curl } \vec{F} = \vec{0}$ in D

$\boxed{\nabla \times \vec{F} = \vec{0}}$ $\Leftrightarrow$ $\vec{F}$ irrotational

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The above definitions are related by the fundamental result: proved below in several parts

The line integral of $\vec{F}$ is path-independent in D if and only if $\vec{F}$ is conservative. In a <u>simply connected</u> domain D, this is equivalent to $\vec{F}$ irrotational.

Let C be a smooth space curve from P to Q, parametrized by $\vec{R}(t)$, $a \le t \le b$.
coordinates $\vec{R}(a)$ $\vec{R}(b)$

Fundamental Theorem of Calculus for Line Integrals of Gradients (Conservative Fields)

Theorem: If the scalar field ϕ is continuously differentiable along C (ie $G(t) = \phi(\vec{R}(t))$ has a continuous derivative) then

$$\boxed{\int_C \nabla\phi \cdot d\vec{R} = \phi(\vec{R}(b)) - \phi(\vec{R}(a))} = \underset{\text{final point}}{\phi(Q)} - \underset{\text{initial}}{\phi(P)}$$

Proof: Rate of change of ϕ along C: $\quad [G(t) = \phi(\vec{R}(t))$

$$\frac{dG}{dt} = \frac{d}{dt}\phi(\vec{R}(t)) = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} = \nabla\phi(\vec{R}(t)) \cdot \vec{R}'(t)$$

Chain rule

Line integral of $\nabla\phi$:

$$\int_C \nabla\phi \cdot d\vec{R} = \int_a^b \underbrace{\nabla\phi(\vec{R}(t)) \cdot \vec{R}'(t)}\, dt$$

$$= \int_a^b \left[\frac{d}{dt}\phi(\vec{R}(t))\right] dt \qquad \left[= \int_a^b G'(t)\,dt = G(b) - G(a)\right.$$

Fundamental Theorem of Calculus

$$= \phi(\vec{R}(t))\Big|_a^b = \phi(\vec{R}(b)) - \phi(\vec{R}(a))$$

$$= \phi(Q) - \phi(P)$$

∎

Note: if C is a regular curve, the union of a finite number of smooth curves, the theorem still holds, treating the line integral as the sum of integrals over disjoint smooth arcs.

Independence of Path for Conservative Fields

Theorem: A continuous vector field $\vec{F}$ in a domain D is conservative if and only if the line integral of $\vec{F}$ is path-independent in D (the line integral along any regular curve C in D depends only on the endpoints P and Q of C.) In this case

$$\int_C \vec{F} \cdot d\vec{R} = \phi(Q) - \phi(P)$$

, where ϕ is the potential for $\vec{F}$.

Proof: 1. "only if"

Assume $\vec{F}$ is conservative i.e. $\vec{F} = \nabla\phi$ in D for some ϕ.

Then along any smooth (or regular) curve C from P to Q,

$$\int_C \vec{F}\cdot d\vec{R} = \int_C \nabla\phi \cdot d\vec{R} = \int_a^b \nabla\phi(\vec{R}(t)) \cdot \vec{R}'(t)\,dt$$

(Fundamental Theorem for Line Integrals)
$$= \phi(\vec{R}(b)) - \phi(\vec{R}(a)) = \phi(Q) - \phi(P)$$

Thus $\int_C \vec{F}\cdot d\vec{R}$ depends only on P and Q, the endpoints, and is independent of the path.

2. "if"

Conversely, assume $\int_C \vec{F}\cdot d\vec{R}$ is path-independent and depends only on the endpoints for any regular curve C in D. We will define a potential function ϕ with the desired properties:

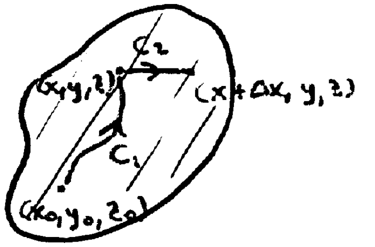
- Fix an arbitrary point (x_0, y_0, z_0) in D: "zero potential".

  For any point (x,y,z) in D, there is an arc C_1 in D from (x_0, y_0, z_0) to (x,y,z) (since D is a domain $\Rightarrow$ connected)

  Define
  $$\phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F}\cdot d\vec{R} \qquad \text{(integrating along } C_1\text{)}$$

  line integral is path-independent $\Rightarrow \phi(x,y,z)$ does not depend on our choice of C_1 $\Rightarrow \phi$ is a well-defined scalar field (single-valued function of (x,y,z))

- Show $\vec{F} = \nabla\phi$:
  $$\frac{\partial\phi}{\partial x} = \lim_{\Delta x\to 0} \frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x}$$



$$\phi(x+\Delta x, y, z) = \int_{(x_0,y_0,z_0)}^{(x+\Delta x,y,z)} \vec{F}\cdot d\vec{R}$$

integrate along C_1 to (x,y,z), then along horizontal C_2 from (x,y,z) to $(x+\Delta x, y, z)$

$$= \underbrace{\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F}\cdot d\vec{R}}_{\phi(x,y,z)} + \int_{(x,y,z)}^{(x+\Delta x,y,z)} \vec{F}\cdot d\vec{R}$$

D is open $\Rightarrow$ an ε-nhd of (x,y,z) is contained in D for some $\varepsilon > 0$. Let $\Delta x < \varepsilon$ $\Rightarrow C_2$ is in D.

$$\Rightarrow \frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{R} \quad \leftarrow \text{along } C_2 : \int_{C_2} \vec{F}\cdot d\vec{R}$$

Along C_2, y and z are constant $\Rightarrow$ $d\vec{R} = dx\, \hat{\imath}$ $\quad (\vec{T} = \hat{\imath}, dy = dz = 0)$
$ds = dx$

$\Rightarrow$ along C_2, $\vec{F}\cdot d\vec{R} = \vec{F}\cdot\hat{\imath}\, dx = F_1\, dx$

$$\Rightarrow \frac{1}{\Delta x}\int_{C_2} \vec{F}\cdot d\vec{R} = \underbrace{\frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x, y, z)} F_1\, dx}_{} = \frac{\Delta x}{\Delta x} F_1(x+\theta\Delta x, y, z), \quad 0 \le \theta \le 1$$

(Mean Value Theorem for Integrals) of a single variable

average of F_1 along C_2, which $\to F_1(x,y,z)$ as $\Delta x \to 0$ since F_1 is continuous $\quad \longrightarrow F_1(x,y,z)$ as $\Delta x \to 0$

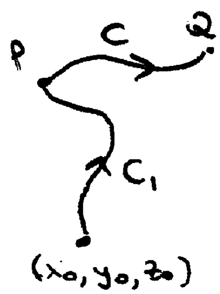
$$\text{ie} \quad \frac{\partial\phi}{\partial x} = \lim_{\Delta x\to 0} \frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x, y, z)} F_1\, dx = F_1(x,y,z)$$

Similarly $\dfrac{\partial\phi}{\partial y} = F_2$, $\dfrac{\partial\phi}{\partial z} = F_3$ $\quad \Rightarrow \vec{F} = \nabla\phi$.

- Show $\int_P^Q \vec{F}\cdot d\vec{R} = \phi(Q) - \phi(P)$ for any P, Q :

Let C be any regular curve from P to Q

C_1 - a curve from (x_0, y_0, z_0) to P.



Then

$$\phi(Q) = \int_{(x_0,y_0,z_0)}^{Q} \vec{F}\cdot d\vec{R} = \int_{C_1} \vec{F}\cdot d\vec{R} + \int_C \vec{F}\cdot d\vec{R}$$

$$= \phi(P) + \int_C \vec{F}\cdot d\vec{R}, \text{ as required.}$$

we can compute $\phi(Q)$ by integrating along C_1, then C (path-independence)

$\boxtimes$

Corollary:

A continuous vector field $\vec{F}$ in a domain D is conservative if and only if $\oint_C \vec{F}\cdot d\vec{R} = 0$ for every regular closed curve C in D

(the circulation around any closed curve is 0.)

Proof: 1. "only if"

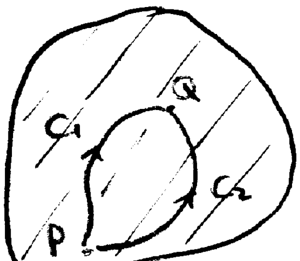
suppose $\vec{F}$ is conservative, $\vec{F} = \nabla\phi$.

If C is a closed curve, its beginning and end coincide, $P=Q$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{R} = \oint_C \nabla\phi \cdot d\vec{R} = \phi(P) - \phi(P) = 0.$$

2. "if"

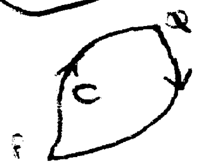
suppose $\oint_C \vec{F}\cdot d\vec{R} = 0$ for every closed curve C.



Consider two points P, Q in D, and let C_1, C_2 be two regular curves from P to Q; by the previous theorem it is sufficient to show the line integral is path-independent, $\int_{C_1} \vec{F}\cdot d\vec{R} = \int_{C_2} \vec{F}\cdot d\vec{R}$

Construct the closed curve $C = C_1 - C_2$

ie going from P to Q along C_1, then back from Q to P along $-C_2$ (C_2 with opposite orientation). Now



$$0 = \oint_C \vec{F}\cdot d\vec{R} = \int_{C_1} \vec{F}\cdot d\vec{R} - \int_{C_2} \vec{F}\cdot d\vec{R} \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{R} = \int_{C_2} \vec{F}\cdot d\vec{R}$$

for any C_1, C_2 from P to Q $\Rightarrow \vec{F}$ is conservative. ☒

---

eg Show that $\vec{F} = (y^2+z)\hat{\imath} + (2xy+z)\hat{\jmath} + (x+\cos z)\hat{k}$ is conservative:

Assume $\vec{F} = \nabla\phi$ ie we need ϕ so $\frac{\partial\phi}{\partial x} = y^2+z$, $\frac{\partial\phi}{\partial y} = 2xy+z$, $\frac{\partial\phi}{\partial z} = x+\cos z$

$$\frac{\partial\phi}{\partial x} = y^2+z \Rightarrow \phi = xy^2 + xz + f_1(y,z) \quad \leftarrow \text{constant w.r.t. } x$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = 2xy + \frac{\partial f_1}{\partial y} \underset{\text{must equal}}{=} 2xy + z \text{, so } \frac{\partial f_1}{\partial y} = z \Rightarrow f_1(y,z) = zy + f_2(z)$$

so we have $\phi = xy^2 + xz + zy + f_2(z)$

$$\Rightarrow \frac{\partial\phi}{\partial z} = x + \frac{\partial f_2}{\partial z} \underset{\text{must equal}}{=} x + \cos z \Rightarrow \frac{\partial f_2}{\partial z} = \cos z \Rightarrow f_2(z) = \sin z + C$$

Thus

$$\phi(x,y,z) = xy^2 + xz + zy + \sin z + C \text{, and clearly } \vec{F} = \nabla\phi.$$

Note: We constructed the potential function explicitly; in a simply connected domain, it is quicker to show $\nabla\times\vec{F} = \vec{0}$ (see later).

eg Show that $\vec{F} = \underbrace{xy}_{F_1}\,\hat{\imath} + \underbrace{x^2(y-2)}_{F_2}\,\hat{\jmath}$ is not conservative:

Method 1: Suppose $\vec{F} = \nabla\phi$: then we need $F_1 = \frac{\partial\phi}{\partial x} = xy$

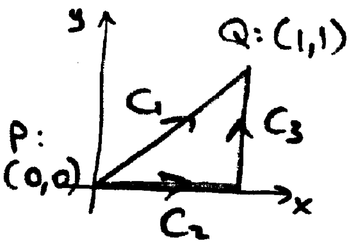
$F_2 = \frac{\partial\phi}{\partial y} = x^2(y-2)$

- but then the mixed partial derivatives would be

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2\phi}{\partial y\,\partial x} = x, \quad \frac{\partial F_2}{\partial x} = \frac{\partial^2\phi}{\partial x\,\partial y} = 2x(y-2)$$

- this contradicts the requirement that $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$, so $\vec{F} = \nabla\phi$ cannot be satisfied by any ϕ.

Method 2: Consider the line integral from $P:(0,0)$ to $Q:(1,1)$ computed two ways:



- Along C_1, $y=x$, $dy = dx$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{R} = \int_{\substack{(0,0)\\ \text{along } C_1}}^{(1,1)} F_1\,dx + F_2\,dy = \int_0^1 \Big[F_1(x,y(x)) + F_2(x,y(x))\frac{dy}{dx}\Big]dx$$

$$= \int_0^1 \left[x^2 + x^2(x-2)\right]dx = \int_0^1 (x^3 - x^2)\,dx$$

$$= \left(\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3\right)\Big|_0^1 = -\tfrac{1}{12}$$

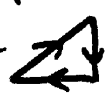
- Along C_2, $y=0$, $dy=0$, so $\int_{C_2} \vec{F}\cdot d\vec{R} = \int_{C_2} F_1\,dx + F_2\,\cancel{dy} = \int_0^1 x\cdot 0\,dx = 0$

- Along C_3, $x=1$, $dx=0$, so $\int_{C_3} \vec{F}\cdot d\vec{R} = \int_0^1 F_2\,dy = \int_0^1 1(y-2)\,dy = \left(\tfrac{1}{2}y^2 - 2y\right)\Big|_0^1 = -\tfrac{3}{2}$

$$\Rightarrow \int_{\substack{(0,0)\\ \text{along } C_2\cup C_3}}^{(1,1)} \vec{F}\cdot d\vec{R} = 0 - \tfrac{3}{2} = -\tfrac{3}{2}$$

- so the line integral from $(0,0)$ to $(1,1)$ is not path-independent $\Rightarrow \vec{F}$ is not conservative.

Equivalently, consider the closed curve $C = C_1 - (C_2\cup C_3)$:

C 

$$\oint_C \vec{F}\cdot d\vec{R} = -\tfrac{1}{12} - \left(0 - \tfrac{3}{2}\right) = \tfrac{3}{2} - \tfrac{1}{12} = \tfrac{17}{12} \neq 0.$$

Method 3: (See later): $\nabla\times\vec{F} = \left[2x(y-2) - x\right]\hat{k} \neq \vec{0}$, but if $\vec{F} = \nabla\phi$, then $\nabla\times\vec{F} = \vec{0}$ - contradiction.

Irrotational Fields :

$\vec{F}$ is irrotational if $\text{curl}\,\vec{F} = \vec{0}$

ie $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$, $\dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}$, $\dfrac{\partial F_2}{\partial z} = \dfrac{\partial F_3}{\partial y}$

Suppose $\vec{F}$ is a conservative vector field in a domain D :

ie $\vec{F} = \nabla\phi$ for some ϕ scalar potential

Then

$$\nabla \times \vec{F} = \nabla \times (\nabla\phi) = \vec{0} \quad (\text{for all } \phi \in C^2(D) : \text{vector identity})$$

$\Rightarrow$ $\boxed{\text{if } \vec{F} \text{ is conservative, then } \vec{F} \text{ is irrotational.}}$

The converse holds in simply connected domains:

Theorem: Let $\vec{F}$ be a vector field defined and continuously differentiable on a simply connected domain D. If $\vec{F}$ is irrotational, $\text{curl}\,\vec{F} = \vec{0}$ in D, then $\vec{F}$ is conservative :

$$\boxed{\nabla \times \vec{F} = \vec{0} \text{ in } D \Rightarrow \exists\, \phi \text{ s.t. } \vec{F} = \nabla\phi \text{ in } D}$$

simply connected

Proof: (For the case $\nabla \times \vec{F} = \vec{0}$ in all space, ie $D = \mathbb{R}^3$) :

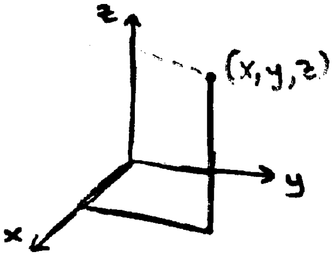
Assume $\vec{F}$ is irrotational in $\mathbb{R}^3$: $\nabla \times \vec{F} = \vec{0}$

To show $\vec{F}$ is conservative, we construct a potential function ϕ for $\vec{F}$ by a line integral, and show $\vec{F} = \nabla\phi$.

Define $\phi(x,y,z) = \displaystyle\int_{(0,0,0)\,C}^{(x,y,z)} \vec{F}\cdot d\vec{R}$ along the path C from $(0,0,0)$ to (x,y,z)

well-defined since $\phi(x,y,z)$ is computed using a specific path C

where C is the curve
(i) From $(0,0,0)$ to $(x,0,0)$ along x-axis ($d\vec{R} = dx\,\hat{i}$)
(ii) From $(x,0,0)$ to $(x,y,0)$ parallel to y-axis ($d\vec{R} = dy\,\hat{j}$)
(iii) From $(x,y,0)$ to (x,y,z) parallel to z-axis ($d\vec{R} = dz\,\hat{k}$)



$$\Rightarrow \phi(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

Show $\nabla\phi = \vec{F}$:

- $\dfrac{\partial\phi}{\partial z} = \dfrac{\partial}{\partial z}\displaystyle\int_0^z F_3(x,y,t)\,dt \underset{\uparrow}{=} F_3(x,y,z)$

Fundamental Theorem of Calculus

$$\left[\text{or } \frac{\partial\phi}{\partial z} = \lim_{\Delta z\to 0} \frac{\overbrace{\phi(x,y,z+\Delta z)}^{}-\phi(x,y,z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_z^{z+\Delta z} F_3(x,y,t)\,dt\right.$$

$\int_0^{z+\Delta z}$ $\qquad\int_0^z$

$$\left. = F_3(x,y,z)\right]$$

- $\dfrac{\partial\phi}{\partial y} = 0 + \dfrac{\partial}{\partial y}\displaystyle\int_0^y F_2(x,t,0)\,dt + \dfrac{\partial}{\partial y}\int_0^z F_3(x,y,t)\,dt$

$$= F_2(x,y,0) + \int_0^z \frac{\partial F_3}{\partial y}(x,y,t)\,dt$$

justify using an argument on interchange of limiting processes: (exchange integral and derivative:

$$\lim_{\Delta y\to 0}\int_0^z \frac{F_3(x,y+\Delta y,t)-F_3(x,y,t)}{\Delta y}\,dt \overset{?}{=} \int_0^z \lim_{\Delta y\to 0}\frac{F_3(x,y+\Delta y,t)-F_3(x,y,t)}{\Delta y}\,dt$$

-okay if $\dfrac{\partial F_3}{\partial y}$ is continuous ...

$\left[\text{but } \nabla\times\vec{F} = \vec{0} \Rightarrow \dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}\right]$

$$\Rightarrow \frac{\partial\phi}{\partial y} = F_2(x,y,0) + \int_0^z \frac{\partial F_2}{\partial t}(x,y,t)\,dt$$

3rd coordinate here is t, not z

$$= F_2(x,y,0) + F_2(x,y,t)\Big|_0^z = \cancel{F_2(x,y,0)} + F_2(x,y,z) - \cancel{F_2(x,y,0)}$$

$$= F_2(x,y,z)$$

- $\dfrac{\partial\phi}{\partial x} = F_1(x,0,0) + \displaystyle\int_0^y \frac{\partial F_2}{\partial x}(x,t,0)\,dt + \int_0^z \frac{\partial F_3}{\partial x}(x,y,t)\,dt$

$\left[\nabla\times\vec{F} = \vec{0} \Rightarrow \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}, \ \dfrac{\partial F_3}{\partial x} = \dfrac{\partial F_1}{\partial z}\right]$

$$= F_1(x,0,0) + \int_0^y \frac{\partial F_1}{\partial y}(x,t,0)\,dt + \int_0^z \frac{\partial F_1}{\partial z}(x,y,t)\,dt$$

$$= \cancel{F_1(x,0,0)} + [\cancel{F_1(x,y,0)} - \cancel{F_1(x,0,0)}] + [F_1(x,y,z) - \cancel{F_1(x,y,0)}]$$

$$= F_1(x,y,z) \quad ; \qquad \Rightarrow \nabla\phi = \vec{F} \qquad \boxtimes$$

The proof can be extended to more general domains

eg for a star-shaped domain D with respect to the point $P:(x_0,y_0,z_0)$, for any point $Q:(x,y,z)$ in D, there is a straight line segment C_Q from P to Q lying wholly inside D. Thus define the potential

$$\phi(x,y,z) = \int_{\substack{(x_0,y_0,z_0)\\ C_Q}}^{(x,y,z)} \vec{F}\cdot d\vec{R} \qquad \text{(integral along } C_Q\text{)}$$

and show, using several vector identities, that $\nabla\phi = \vec{F}$. (see text)

For a general simply connected domain, this theorem follows from <u>Stokes' Theorem</u> (later).

---

eg show that $\vec{F} = (y^2+z)\hat{\imath} + (2xy+2)\hat{\jmath} + (x+\cos z)\hat{k}$ is conservative, and find the potential ϕ: (compare p. 125)

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+z & 2xy+2 & x+\cos z \end{vmatrix} = 0\,\hat{\imath} + (1-1)\hat{\jmath} + (2y-2y)\hat{k} = \vec{0} \ (\text{in } \mathbb{R}^3)$$

$\Rightarrow \vec{F}$ is conservative

Using the above formula (the method of p. 125 is usually preferable)

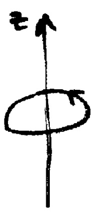
$$\phi(x,y,z)-\phi(0,0,0) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

$$= \int_0^x 0\,dt + \int_0^y (2xt+2)\,dt + \int_0^z (x+\cos t)\,dt$$

$$= 0 + xy^2 + 2y + xz + \sin z \quad \text{, as before}$$

---

eg consider $\vec{F} = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2} = -\dfrac{\sin\theta}{\rho}\hat{\imath} + \dfrac{\cos\theta}{\rho}\hat{\jmath} = \dfrac{1}{\rho}\hat{e}_\theta$ (cylindrical coordinates)



$[\Rightarrow F_\rho = 0,\ F_\theta = \frac{1}{\rho},\ F_z = 0]$

$\Rightarrow \operatorname{curl}\vec{F} = 0\,\hat{e}_\rho + 0\,\hat{e}_\theta + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{1}{\rho}\right)\hat{e}_z = \vec{0} \Rightarrow \vec{F}$ is <u>irrotational</u>

<u>But</u> consider a circle, radius about z axis, in x-y plane: Parametrize $\rho=a,\ \theta=t,\ z=0$ $(d\theta/dt=1)$

$\vec{R}(t) = a\,\hat{e}_\rho \quad (0\le t\le 2\pi) \Rightarrow \vec{R}'(t) = a\,\hat{e}_\theta$

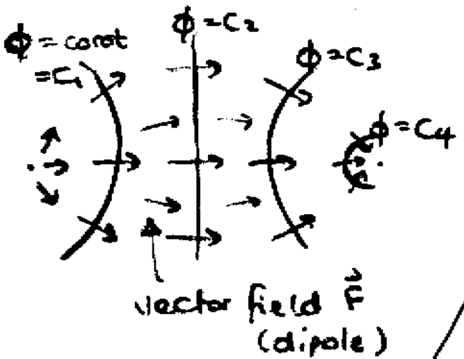
$$\oint_C \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \frac{1}{a}\hat{e}_\theta\cdot a\,\hat{e}_\theta\,dt = \int_0^{2\pi} dt = 2\pi \neq 0 \ \text{(for all } a\text{)}$$

$\Rightarrow \vec{F}$ is <u>not</u> conservative! No contradiction, since the domain is not simply connected ($\vec{F}$ is not defined on z-axis $\rho=0$)

Interpretation of the scalar potential ϕ:

$\vec{F} = \nabla\phi$ $\Rightarrow$ $\vec{F}$ is normal to the level surfaces ϕ = constant

$\Rightarrow$ flow lines of the vector field $\vec{F}$ = grad ϕ are orthogonal to its equipotential surfaces



Application: conservative forces

it is common to write $\vec{F} = -\nabla\phi$ ← opposite sign for ϕ

then ϕ is interpreted as potential energy

$\vec{F}$ points in direction of greatest decrease of potential energy.

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eg Electrostatics: Maxwell's equations for time-independent electric fields $\vec{E}$ in free space:

$\vec{E}$: electric field
ρ: charge density

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \quad , \quad \nabla\times\vec{E} = \vec{0} \leftarrow \text{time-indep: } \frac{\partial\vec{B}}{\partial t} = \vec{0}$$

In a simply connected domain, this implies $\vec{E} = \underset{\text{by convention}}{-}\nabla\phi = -\nabla V$

ϕ: electric potential

The scalar potential satisfies Poisson's equation: $\nabla^2\phi = -\rho/\varepsilon_0$

Summary: In a simply connected domain D, the following are equivalent for a smooth vector field $\vec{F}$ on D:

i) $\vec{F}$ is conservative ie $\vec{F} = \nabla\phi$ for some ϕ: scalar potential

ii) The line integral of $\vec{F}$ is path-independent ie for any two regular paths C_1, C_2 from point P to Q in D, $\int_{C_1}\vec{F}\cdot d\vec{R} = \int_{C_2}\vec{F}\cdot d\vec{R}$

iii) The line integral around any regular closed curve C in D vanishes; ie $\oint_C \vec{F}\cdot d\vec{R} = 0$

iv) $\vec{F}$ is irrotational ie $\nabla\times\vec{F} = \vec{0}$.

If D is connected, not simply connected, property iv) does not imply i), ii), iii)

Solenoidal Fields

As usual, let $\vec{F}$ be a continuously differentiable vector field in a domain D.

Def. A vector field $\vec{F}$ is solenoidal in a domain D if div $\vec{F} = 0$ in D.

$$\boxed{\nabla \cdot \vec{F} = 0} \Leftrightarrow \vec{F} \text{ solenoidal} \quad \text{(divergence-free)}$$

Def: If there exists a smooth vector field $\vec{G}$ on D so that $\vec{F} = \nabla \times \vec{G}$, then $\vec{G}$ is called a vector potential for $\vec{F}$

$$\boxed{\vec{F} = \nabla \times \vec{G}} \leftarrow \text{vector potential}$$

Notation: Usually use $\vec{A}$ for vector potential

Suppose $\vec{F} = \text{curl } \vec{G}$ in D for some vector potential $\vec{G}$ ← components $C^2(D)$

Then

$$\nabla \cdot \vec{F} = \nabla \cdot (\nabla \times \vec{G}) = 0 \qquad \text{(vector identity)}$$

ie $\vec{F}$ is solenoidal : the curl of a vector field is a solenoidal vector field.

→ $\boxed{\text{if } \vec{F} \text{ has a vector potential } \vec{G}, \text{ then } \vec{F} \text{ is solenoidal}}$

Recall: we had: if $\vec{F}$ has a scalar potential ϕ (ie $\vec{F} = \nabla\phi$: conservative) then $\vec{F}$ is irrotational, $\nabla \times \vec{F} = \vec{0}$.

Partial converse, for simply connected domains: if $\vec{F}$ is irrotational in a simply connected domain D, then $\vec{F}$ is conservative ie $\exists\ \phi$ s.t. $\vec{F} = \nabla\phi$

The converse holds in simply connected domains: ← assume star-shaped

Theorem: Let $\vec{F}$ be a vector field defined and continuously differentiable on a simply connected domain D. If $\vec{F}$ is solenoidal, div $\vec{F} = 0$ in D, then there exists a vector potential $\vec{G}$ for $\vec{F}$ in D:

$$\boxed{\nabla \cdot \vec{F} = 0 \text{ in } D \Rightarrow \exists \vec{G} \text{ s.t. } \vec{F} = \nabla \times \vec{G} \text{ in } \underset{\text{simply connected}}{D}}$$

Proof: • (For the case $\nabla\cdot\vec{F}=0$ in all space i.e. $D=\mathbb{R}^3$)

Assume $\vec{F}$ is solenoidal in $\mathbb{R}^3$: $\nabla\cdot\vec{F}=0$

We show the existence of a vector potential $\vec{G}$ by construction :

Define
$$\vec{G}(x,y,z)=\left[\int_0^z F_2(x,y,t)\,dt-\int_0^y F_3(x,t,0)\,dt\right]\hat{\imath}$$
$$+\left[-\int_0^z F_1(x,y,t)\,dt\right]\hat{\jmath}\qquad+0\,\hat{k}$$

Then
$$\nabla\times\vec{G}=\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z}\right)\hat{\imath}+\left(\frac{\partial G_1}{\partial z}-\frac{\partial G_3}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)\hat{k}$$

$$=F_1(x,y,z)\,\hat{\imath}+F_2(x,y,z)\,\hat{\jmath}$$
$$+\Big[\underbrace{-\int_0^z\frac{\partial F_1}{\partial x}(x,y,t)\,dt-\int_0^z\frac{\partial F_2}{\partial y}(x,y,t)\,dt+F_3(x,y,0)}_{\text{need to show that this is } F_3(x,y,z)}\Big]\hat{k}$$

but $\nabla\cdot\vec{F}=0$ ie $\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=0$

so
$$[\nabla\times\vec{G}]_3=\int_0^z\left[-\frac{\partial F_1}{\partial x}-\frac{\partial F_2}{\partial y}\right](x,y,t)\,dt+F_3(x,y,0)$$

3rd coordinate here is t
$$=\int_0^z\frac{\partial F_3}{\partial t}(x,y,t)\,dt+F_3(x,y,0)$$

$$=\left[F_3(x,y,z)-\cancel{F_3(x,y,0)}\right]+\cancel{F_3(x,y,0)}=F_3(x,y,z)$$

$$\Rightarrow\ \nabla\times\vec{G}=\vec{F}\ ,\quad\text{as required.}$$

☒

• (for star-shaped domains)

Let D be star-shaped w.r.t. the point $P:(x_0,y_0,z_0)$ ($\vec{R}_0$) ; we wish to find $\vec{G}$ at any point $Q:(x,y,z)$ ($\vec{R}$) in D. Since D is star-shaped, the line segment C_{PQ} : $\vec{r}(t)=\vec{R}_0+t(\vec{R}-\vec{R}_0)$, $0\le t\le 1$ lies entirely in D. Define $\vec{G}(x,y,z)=t\int_{C_{PQ}}\vec{F}\times d\vec{r}=\int_0^1 t\left(\vec{F}\times\frac{d\vec{r}}{dt}\right)dt$

Then $\nabla\times\vec{G}=\vec{F}$. (optional proof below...)

$\nabla\cdot\vec{F}=0 \Rightarrow \vec{F}=\nabla\times\vec{G}$ for a vector field $\vec{G}$: (optional)

Proof for star-shaped domains :

Let D be star-shaped w.r.t. the point $P:(x_0,y_0,z_0)$ with position vector $\vec{R}_0$. We need to find the vector potential $\vec{G}$ at any point $Q:(x,y,z)$ (position vector $\vec{R}$) in D.

Since D is star-shaped, the line segment from P to Q:

$$C_{PQ}: \quad \vec{r}(t) = \vec{R}_0 + \underset{\text{parameter}}{t}(\vec{R}-\vec{R}_0), \quad 0 \le t \le 1$$

lies entirely in D.

Define
$$\vec{G}(x,y,z) = t\int_{C_{PQ}} \vec{F}\times d\vec{r} = \int_0^1 t\left(\vec{F}\times\frac{d\vec{r}}{dt}\right)dt$$

where $\vec{F}$ is a function of the parameter t via $\vec{F}=\vec{F}(\vec{r}(t)) = \vec{F}(\vec{R}_0+t(\vec{R}-\vec{R}_0))$;

explicitly
$$\vec{F} = \vec{F}(\underbrace{x_0+t(x-x_0)}_{X}, \underbrace{y_0+t(y-y_0)}_{Y}, \underbrace{z_0+t(z-z_0)}_{Z}) = \vec{F}(X,Y,Z)$$

We need to show that $\nabla\times\vec{G}=\vec{F}$.

The operator $\nabla = \hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$ acts on the variables x,y,z (corresponding to $\vec{R}$ at the endpoint of the integral) and is independent of t, so we can bring $\nabla\times$ inside the integral:

$$\nabla\times\vec{G} = \nabla\times\int_0^1 t\left(\vec{F}\times\frac{d\vec{r}}{dt}\right)dt = \int_0^1 t\,\nabla\times\left(\vec{F}\times\frac{d\vec{r}}{dt}\right)dt$$

vector identity! (xiv), p.76
$$= \int_0^1 t\left[\left(\frac{d\vec{r}}{dt}\cdot\nabla\right)\vec{F} - (\vec{F}\cdot\nabla)\frac{d\vec{r}}{dt} + \left(\nabla\cdot\frac{d\vec{r}}{dt}\right)\vec{F} - (\nabla\cdot\vec{F})\frac{d\vec{r}}{dt}\right]dt$$

Now $\frac{d\vec{r}}{dt} = \vec{R}-\vec{R}_0 \Rightarrow \nabla\cdot\frac{d\vec{r}}{dt} = \nabla\cdot(\vec{R}-\vec{R}_0) = \nabla\cdot\vec{R} = 3$ $\quad\left[\nabla\cdot\vec{R} = \partial_i x_i = \delta_{ii} = 3\right.$

$(\vec{F}\cdot\nabla)\frac{d\vec{r}}{dt} = \vec{F}\cdot\nabla(\vec{R}-\vec{R}_0) = \vec{F}$ $\quad\left[[(\vec{F}\cdot\nabla)\vec{R}]_j = (F_i\partial_i)(\vec{R})_j = F_i\partial_i x_j = F_i\delta_{ij} = F_j\right.$

In evaluating ∇ acting on $\vec{F}$, we need to be a bit careful:

∇ contains derivatives w.r.t. x,y,z, while $\vec{F}$ is a function of X,Y,Z, with $X = x_0+t(x-x_0)$ etc.

Thus for instance

$$\frac{\partial \vec F}{\partial x} = \frac{\partial}{\partial x}\vec F(\underbrace{x_0+t(x-x_0)}_{X}, \underbrace{y_0+t(y-y_0)}_{Y}, \underbrace{z_0+t(z-z_0)}_{Z})$$

chain rule $= \dfrac{\partial \vec F}{\partial X}\dfrac{\partial X}{\partial x} = t\dfrac{\partial \vec F}{\partial X}$ and similarly $\dfrac{\partial \vec F}{\partial y} = t\dfrac{\partial \vec F}{\partial Y}$, $\dfrac{\partial \vec F}{\partial z} = t\dfrac{\partial \vec F}{\partial Z}$

That is, ∇ acts on the components of $\vec F$ via $\nabla^* = \underbrace{\hat\imath\frac{\partial}{\partial X} + \hat\jmath\frac{\partial}{\partial Y} + \hat k\frac{\partial}{\partial Z}}_{\text{derivatives wrt arguments of } \vec F}$

(along the line segment (PQ))

Hence $\nabla\cdot\vec F = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z} = t\left[\dfrac{\partial F_1}{\partial X} + \dfrac{\partial F_2}{\partial Y} + \dfrac{\partial F_3}{\partial Z}\right] = t\,\nabla^*\cdot\vec F = t\,\mathrm{div}\,\vec F = 0$

since by assumption, $\vec F$ is solenoidal (divergence-free).

Combining our calculations so far into our expression for $\nabla\times\vec G$:

$$\nabla\times\vec G = \int_0^1 t\left[\left(\frac{d\vec r}{dt}\cdot\nabla\right)\vec F - \vec F + 3\vec F - 0\right]dt = \int_0^1\left[2t\vec F + t\left(\frac{d\vec r}{dt}\cdot\nabla\right)\vec F\right]dt$$

This expression can be simplified by considering the derivative of $t^2\vec F(\vec r(t))$ along the line segment, and using the chain rule and product rule:

$$\frac{d}{dt}(t^2\vec F) = \frac{d}{dt}\left[t^2\vec F(\vec r(t))\right] = 2t\vec F(\vec r(t)) + t^2\frac{d}{dt}\vec F(\vec r(t))$$

$$= 2t\vec F + t^2\left(\frac{d\vec r}{dt}\cdot\nabla^*\right)\vec F(\vec r(t))$$ wrt components X, Y, Z of $\vec F$

$$= 2t\vec F + t\left(\frac{d\vec r}{dt}\cdot\nabla\right)\vec F$$ ← this expression appears in the integral for $\nabla\times\vec G$.

[more explicitly, in terms of components:

$$t^2\frac{d}{dt}\vec F(\vec r(t)) = t^2\frac{d}{dt}\vec F(x_0+t(x-x_0), y_0+t(y-y_0), z_0+t(z-z_0))$$

$$= t^2\left[\frac{\partial\vec F}{\partial X}\frac{dX}{dt} + \frac{\partial\vec F}{\partial Y}\frac{dY}{dt} + \frac{\partial\vec F}{\partial Z}\frac{dZ}{dt}\right] = t^2\left[(x-x_0)\frac{\partial\vec F}{\partial X} + (y-y_0)\frac{\partial\vec F}{\partial Y} + (z-z_0)\frac{\partial\vec F}{\partial Z}\right]$$

but $\dfrac{\partial\vec F}{\partial X} = \dfrac{1}{t}\dfrac{\partial\vec F}{\partial x}$, etc

$$= t\left[(x-x_0)\frac{\partial\vec F}{\partial x} + (y-y_0)\frac{\partial\vec F}{\partial y} + (z-z_0)\frac{\partial\vec F}{\partial z}\right]$$

$$= t\left[(\vec R-\vec R_0)\cdot\nabla\right]\vec F = t\left(\frac{d\vec r}{dt}\cdot\nabla\right)\vec F$$]

Substituting, we thus have

$$\nabla\times\vec G = \int_0^1\frac{d}{dt}\left(t^2\vec F(\vec r(t))\right)dt = t^2\vec F(\vec r(t))\Big|_0^1 = \vec F(\vec r(1)) = \vec F(\vec R) = \vec F(x,y,z)$$

as required. ∎

(end of optional proof)

Note: $\nabla \times (\nabla \psi) = \vec{0}$ for any scalar field ψ $[\in C^2(D)]$

$\Rightarrow$ if $\vec{G}$ is a vector potential for $\vec{F}$, then so is $\vec{G} + \nabla\psi$ for any ψ:

$$\vec{F} = \nabla \times \vec{G} = \nabla \times (\vec{G} + \nabla\psi)$$

ie the vector potential is determined only up to a gradient.

Gauge invariance: the freedom in the choice of the gradient term $\nabla\psi$ in the vector potential (the physics is independent of the choice of ψ) [a fundamental principle eg "gauge field theories"]

eg fix the value of $\nabla \cdot \vec{G}$

(can be chosen independently of $\nabla \times \vec{G}$)

Comment on degrees of freedom:

A general vector field $\vec{F}$ has three independent components (3 degrees of freedom)

Irrotational $\nabla \times \vec{F} = \vec{0}$

- The irrotational (curl-free) condition $\nabla \times \vec{F} = \vec{0}$ appears to impose three conditions on the components of $\vec{F}$

  but the components of any curl are not independent ($\nabla \cdot \nabla \times \vec{F} = \vec{0}$, all $\vec{F}$)

  $\Rightarrow$ if we know that the first two components of $\nabla \times \vec{F}$ are zero, the third component is automatically constrained (to be z-independent)

  $\Rightarrow$ the condition $\nabla \times \vec{F} = \vec{0}$ is essentially two independent constraints, leaving one degree of freedom contained in ϕ:

  $$\vec{F} = \nabla\phi, \quad \phi \text{ unconstrained}$$

  [$\Rightarrow$ it is usually convenient to reformulate a problem i.t.o. ϕ — need to find only one function]

Solenoidal $\nabla \cdot \vec{F} = 0$

- The solenoidal (divergence-free) condition $\nabla \cdot \vec{F} = 0$ imposes one constraint $\Rightarrow$ two degrees of freedom left.

The vector potential $\vec{G}$ appears to have three independent components, but the gauge freedom $\vec{F} = \nabla \times \vec{G} = \nabla \times [\vec{G} + \nabla\psi]$ renders one component superfluous

[eg by choosing $\frac{\partial \psi}{\partial z} = -G_3(x,y,z)$, we can cancel the z-component of $\vec{G}$]

eg $\vec{F}(x,y,z) = \vec{A} = \text{constant}$

$$\Rightarrow \nabla\cdot\vec{F} = \nabla\cdot\vec{A} = 0 \,, \qquad \nabla\times\vec{F} = \vec{0}$$

Vector potential $\vec{G} = \int_0^1 t\left(\vec{F}\times\frac{d\vec{r}}{dt}\right)dt$ ← integrate along straight line from origin to $\vec{R} = x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{r}(t) = \vec{R}_0 + t(\vec{R}-\vec{R}_0) = t\vec{R}$

indep. of t

$$= \int_0^1 t(\vec{A}\times\vec{R})\,dt \qquad \Rightarrow \frac{d\vec{r}}{dt} = \vec{R}$$

$$= (\vec{A}\times\vec{R})\int_0^1 t\,dt = \tfrac{1}{2}(\vec{A}\times\vec{R})$$

Scalar potential $\phi = \int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_0^1 \vec{A}\cdot\vec{R}\,dt = \vec{A}\cdot\vec{R}$

$$\Rightarrow \vec{A} = \nabla(\vec{A}\cdot\vec{R}) = \nabla\times\left(\tfrac{1}{2}\vec{A}\times\vec{R}\right) \qquad \text{(check)}$$

Recall: • angular velocity field $\vec{\omega}$ ⇒ (linear) velocity $\vec{v} = \vec{\omega}\times\vec{R}$

⇒ $\vec{G} = \frac{1}{2}\vec{A}\times\vec{R}$ is the velocity field corresponding to constant angular velocity $\frac{1}{2}\vec{A}$

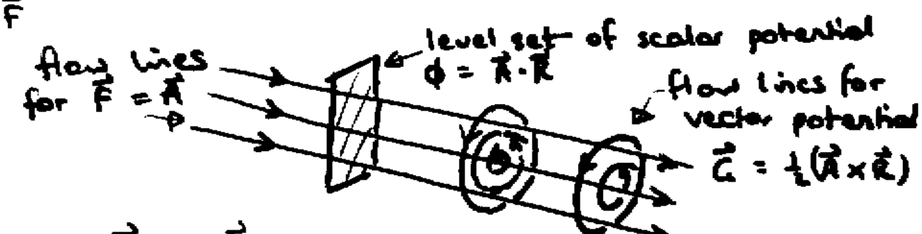
• in general, if $\vec{G} = \vec{v}$ is a velocity field, then $\nabla\times\vec{G} = \nabla\times\vec{v}$ is twice the local angular velocity vector field.

vorticity

Interpretation of vector potential:

- Given a diagram of the vector potential field $\vec{G}$, one can sketch its curl: the flow lines of $\vec{F} = \nabla\times\vec{G}$ are the vortex lines of $\vec{G}$
- Conversely, we can visualize the flow lines of $\vec{G}$ as "wrapped around" the flow lines of $\vec{F}$

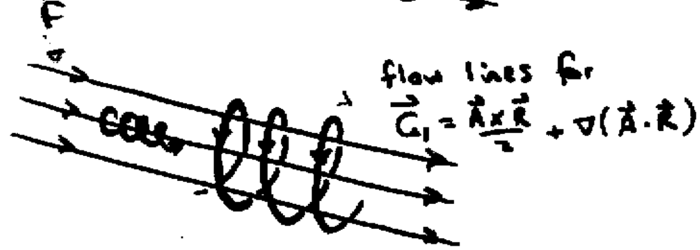
eg $\vec{F} = \vec{A} = \text{const.}$



We can add any gradient to $\vec{G}$

eg $\vec{G}_1 = \vec{G} + \nabla(\vec{A}\cdot\vec{R}) = \frac{1}{2}(\vec{A}\times\vec{R}) + \vec{A}$

- in this case the flow lines of $\vec{G}_1$ acquire a "downstream component"



Oriented Surfaces

- eg graphs of functions in the plane : $z = f(x,y)$
  (too restrictive in general eg sphere, torus cannot be represented as graphs)
- eg level set of a scalar field, $g(x,y,z) = \text{constant}$
  eg a graph is given by $g(x,y,z) = z - f(x,y) = 0$
- more generally, (roughly speaking) a surface is a union of disjoint two-dimensional pieces, each obtained by deforming – rolling, stretching, bending – a region of the plane, and gluing the pieces...

Basic property of surfaces locally: direction of normal vector at each point

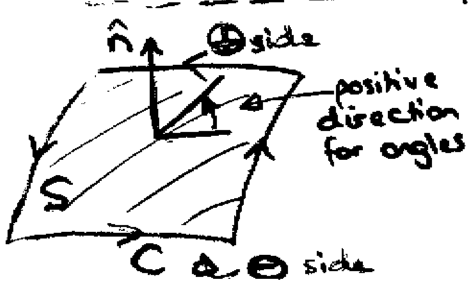
eg level set $g(x,y,z) = \text{const}$: Normal to surface is ∇g
- two unit normals $\hat{n} = +\frac{\nabla g}{|\nabla g|}$, $\hat{n} = -\frac{\nabla g}{|\nabla g|}$

The surface S is smooth if we can choose a unit normal $\hat{n}$ at each point of S so that $\hat{n}$ varies continuously on S eg sphere

S is piecewise smooth if it consists of a finite number of smooth pieces "joined together" eg cube

Orient the surface by choosing $\hat{n}$ (eg choose either $+\frac{\nabla g}{|\nabla g|}$ or $-\frac{\nabla g}{|\nabla g|}$)
$\Rightarrow$ a consistent (continuous) field of unit normal vectors
- gives a "positive" $\oplus$ side and a "negative" $\ominus$ side
(a surface S has two possible orientations)

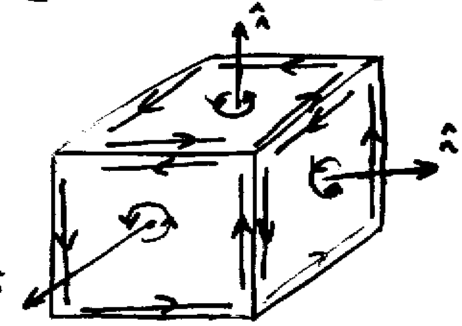
Bounded surface – if S is bounded by a regular closed curve C



Convention: the orientation of S follows from the orientation of C by the right-hand rule.
[curl fingers of right hand in direction of C
$\Rightarrow$ thumb points in positive direction, of $\hat{n}$]

A <u>closed surface</u> has no boundary (encloses a region of space)

<u>Convention</u>: choose $\hat{n}$ to point <u>out</u> of S, away from the enclosed region: <u>outward normal</u>



A surface is <u>orientable</u> if it has two sides ie at each point of S, there is a "positive" and a "negative" side
"outside" "inside" ← for closed surfaces

Some surfaces are non-orientable eg Möbius strip
– we consider only orientable surfaces.

We wish to define surface integrals in a similar way to the previous definition of line integrals.

Recall: <u>Line integrals</u>: along a smooth curve C: $a \le t \le b$

We needed:
- Parametrization of C: $\vec{R} = \vec{R}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$
- Local tangent i.t.o. parametrization: $\vec{T} = \frac{d\vec{R}}{dt} \Big/ \left|\frac{d\vec{R}}{dt}\right| = \frac{d\vec{R}}{ds}$
- Element of arc length: $ds = |d\vec{R}| = \left|\frac{d\vec{R}}{dt}\right| dt$
- ⇒ line element $d\vec{R} = \vec{T}\,ds = \frac{d\vec{R}}{dt}\,dt = \hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz$ $= \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) dt$

A <u>space curve</u> is <u>one</u>-dimensional: need a single parameter t

- Hence define the line integrals $\int_C f\,ds = \int_a^b f(\vec{R}(t)) \left|\frac{d\vec{R}}{dt}\right| dt$,
$\int_C \vec{F} \cdot d\vec{R} = \int_C \vec{F} \cdot \vec{T}\,ds = \int_a^b \vec{F}(\vec{R}(t)) \cdot \frac{d\vec{R}}{dt}\,dt$

Similarly <u>Surface integrals</u>: over a smooth surface S:

We need:
- Parametrization of S: $\vec{R} = \vec{R}(u,v)$
- Local normal $\hat{n}$ i.t.o. parametrization: $\hat{n}(u,v) \ldots$
- Surface element $d\vec{S} = \hat{n}\,dS$, element of surface area $dS = |d\vec{S}|$
- Hence define surface integrals $\iint_S f\,dS$, $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\,dS$

A <u>surface</u> is <u>two</u>-dimensional

Parametrized Surfaces

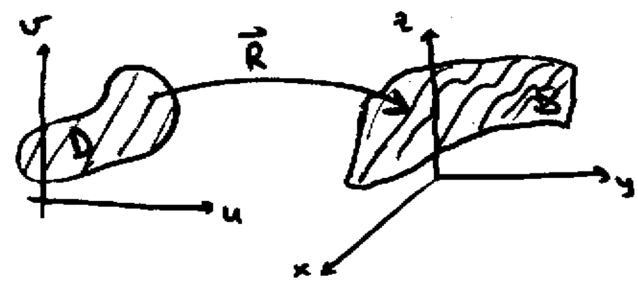
A surface in space ($\mathbb{R}^3$) is a two-dimensional object

$\Rightarrow$ need two parameters (eg u,v) to describe it : $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ parametric representation of surface

position vector

$$\Rightarrow \quad \vec{R} = \vec{R}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$$

$\Rightarrow$ a parametrized surface is given by a function (map) $\vec{R} : D \to \mathbb{R}^3$, where D is some domain in $\mathbb{R}^2$. The surface $S = \vec{R}(D)$ is the image of the domain D under $\vec{R}$.



(similarly, we can define surfaces embedded in $\mathbb{R}^n$: $\vec{R} : D \to \mathbb{R}^n$, $\vec{R}(u,v) \in \mathbb{R}^n$)

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eg <u>plane</u> through $\vec{R}_0$, parallel to independent vectors $\vec{A}, \vec{B}$:

$$\vec{R} = \vec{R}_0 + s\vec{A} + t\vec{B} \qquad (s,t : \text{parameters})$$

Alternatively, given the non-parametric equation for a plane,

$$ax + by + cz = d \quad (\text{with normal } \vec{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}, \text{ and } \vec{R}_0 \cdot \vec{n} = d\ )$$

we can parametrize it (for $c \neq 0$) by $x = u,\ y = v,\ z = \dfrac{d - au - bv}{c}$

eg <u>sphere</u>, centre (x_0, y_0, z_0), radius a :

$$x = x_0 + a\sin\phi\cos\theta, \quad y = y_0 + a\sin\phi\sin\theta, \quad z = z_0 + a\cos\phi$$

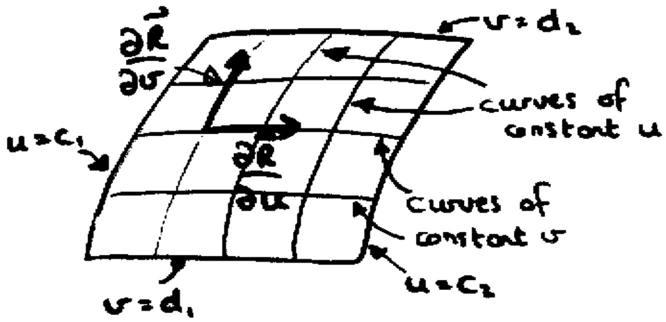
(parameters ϕ, θ)

As the parameters u and v vary, the tip of the position vector $\vec{R}(u,v)$ traces out the surface.

If we fix $v = v_0$, and vary u only, the position vector $\vec{R}(u, v_0)$ traces a curve lying in the surface, with tangent $\frac{\partial \vec{R}}{\partial u}$ ← keep v fixed

$$\frac{\partial \vec{R}}{\partial u} = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k}$$: tangent to curve $v = \text{constant}$

Similarly, $$\frac{\partial \vec{R}}{\partial v} = \frac{\partial x}{\partial v}\hat{\imath} + \frac{\partial y}{\partial v}\hat{\jmath} + \frac{\partial z}{\partial v}\hat{k}$$: tangent to curve $u = \text{constant}$



Surface $S = \vec{R}(u,v)$:
$c_1 \le u \le c_2, \; d_1 \le v \le d_2$

Assume $\frac{\partial \vec{R}}{\partial u}$ and $\frac{\partial \vec{R}}{\partial v}$ exist at each point on S (ie for each $(u_0, v_0) \in D$), and are nonzero and non-parallel at each point; also assume $\frac{\partial \vec{R}}{\partial u}, \frac{\partial \vec{R}}{\partial v}$ are continuous on the surface (ie each component is a continuous function on D)

– this defines a smooth surface and a continuously differentiable parametrization.

$\frac{\partial \vec{R}}{\partial u}, \frac{\partial \vec{R}}{\partial v}$ are tangent to curves on the surface ⇒ tangent to the surface

⇒ $\boxed{\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \text{ is normal to the surface } S.}$

Assume that the parameters u, v are labelled so that $\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}$ points in the positive direction on S

– else interchange u, v

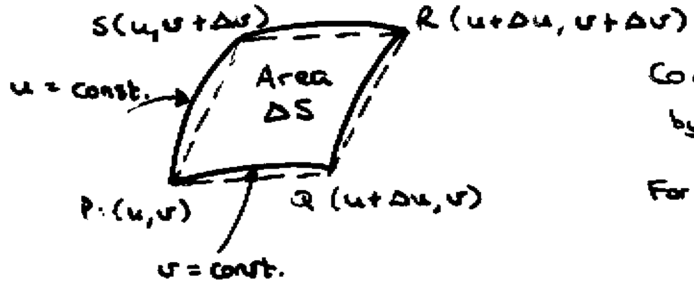
Normal to surface $\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}$ $\left(= \vec{N}(u,v) \right)$

⇒ unit normal $$\boxed{\hat{n} = \frac{\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}}{\left| \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right|}} \quad \left(= \frac{\vec{N}}{|\vec{N}|} \right)$$

(careful: avoid confusing this $\vec{N}$ with the principal normal of a space curve)

Regular Surface Element

(a heuristic argument...)



Consider a small patch of surface bounded by curves of constant u and v.

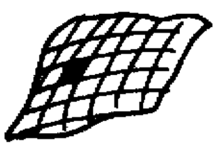
For small Δu, Δv, approximate the patch by a parallelogram:

$$\vec{PQ} = \vec{R}(u+\Delta u, v) - \vec{R}(u,v) \approx \frac{\partial \vec{R}}{\partial u}\Delta u$$

$$\vec{PS} = \vec{R}(u, v+\Delta v) - \vec{R}(u,v) \approx \frac{\partial \vec{R}}{\partial v}\Delta v$$

Area of parallelogram: given by cross product magnitude:

$$\text{Area } \Delta S \approx |\vec{PQ} \times \vec{PS}| \approx \left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right| \Delta u\, \Delta v$$



Get surface area of a surface by summing areas of patches, let Δu, $\Delta v \to 0$:

$$\text{Area} \approx \sum_{i=1}^{N} \Delta S_i \approx \sum_{i=1}^{N} \left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right|_i \Delta u\, \Delta v$$

$N \to \infty$, $\Delta u, \Delta v \to 0$

of a smooth parametric surface → $\boxed{\text{Surface area } S = \iint_D \left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right| du\, dv}$ ← parametric form

Define surface element $\boxed{d\vec{S} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv} = \frac{\partial \vec{R}}{\partial u} du \times \frac{\partial \vec{R}}{\partial v} dv$

- direction of vector $d\vec{S}$ is normal to the surface S (in positive direction)
- magnitude of $d\vec{S}$ is element of area dS.

element of area $\boxed{dS = |d\vec{S}| = \left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right| du\, dv}$

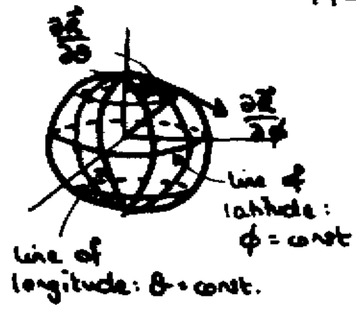
Compare for space curves: $ds = |d\vec{r}|$, $\vec{T} = \frac{d\vec{R}}{dt} \Big/ \left|\frac{d\vec{R}}{dt}\right| = \frac{d\vec{R}}{ds}$, $d\vec{R} = \vec{T}ds$

unit normal $\hat{n} = \dfrac{\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}}{\left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right|} = \dfrac{d\vec{S}}{dS} \Rightarrow \boxed{d\vec{S} = \hat{n}\, dS}$

$$\Rightarrow \boxed{\text{Surface area} = \iint_S dS = \iint_S |d\vec{S}| = \iint_S \hat{n} \cdot d\vec{S}}$$

eg surface area of sphere, radius a:

$$x = a\sin\phi\cos\theta, \quad y = a\sin\phi\sin\theta, \quad z = a\cos\phi$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi \quad \text{(parameters } \phi, \theta)$$



$$\frac{\partial \vec{R}}{\partial \phi} = a\cos\phi\cos\theta\,\hat{i} + a\cos\phi\sin\theta\,\hat{j} - a\sin\phi\,\hat{k}$$

("South" along lines of longitude, tangent to ϕ coordinate curves)

$$\frac{\partial \vec{R}}{\partial \theta} = -a\sin\phi\sin\theta\,\hat{i} + a\sin\phi\cos\theta\,\hat{j}$$

("East" along lines of latitude)

ϕ, θ, r form a right-handed system $\Rightarrow \frac{\partial \vec{R}}{\partial \phi} \times \frac{\partial \vec{R}}{\partial \theta}$ points out of sphere, in direction of increasing r

$$\frac{\partial \vec{R}}{\partial \phi} \times \frac{\partial \vec{R}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\phi\cos\theta)\,\hat{i} + (a^2\sin^2\phi\sin\theta)\,\hat{j} + (a^2\sin\phi\cos\phi)\,\hat{k}$$

$(= a\sin\phi\,(x\hat{i} + y\hat{j} + z\hat{k}) = a\sin\phi\,\vec{R}$
– in outward radial direction, as expected)

$$dS = \left|\frac{\partial \vec{R}}{\partial \phi} \times \frac{\partial \vec{R}}{\partial \theta}\right| d\phi\, d\theta = a^2\sin\phi\,\left|\sin\phi\cos\theta\,\hat{i} + \sin\phi\sin\theta\,\hat{j} + \cos\phi\,\hat{k}\right| d\phi\, d\theta$$
$$= a^2\sin\phi\,\left[\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi\right]^{1/2} d\phi\, d\theta$$
$$= a^2\sin\phi\, d\phi\, d\theta$$

as derived previously

Surface area $$S = \iint_{\text{sphere}} dS = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta = a^2 \cdot 2\pi \cdot (-\cos\phi)\Big|_0^{\pi}$$
$$= 4\pi a^2$$
(as expected)

---

eg change of variables for double integrals:

A coordinate transformation in the plane

$$x = x(u,v), \quad y = y(u,v)$$

can be thought of as a parametrized surface with no z component: $z(u,v) = 0$

$$\vec{R}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} \quad \Rightarrow \quad \frac{\partial \vec{R}}{\partial u} = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}, \quad \frac{\partial \vec{R}}{\partial v} = \frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}$$

$$\Rightarrow \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\hat{k} = \det \underbrace{\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}}_{\text{Jacobian matrix}} \hat{k} = \underbrace{\det \frac{\partial(x,y)}{\partial(u,v)}}_{\text{Jacobian determinant}}\, \hat{k}$$

$\Rightarrow$ Area element $\overset{dx\,dy}{dS} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$

Special case: <u>Graphs</u> of functions of two variables

$$z = f(x,y)$$

Use x, y as parameters, $(x,y) \in D \subset \mathbb{R}^2$

$$\vec{R}(x,y) = x\,\hat{i} + y\,\hat{j} + f(x,y)\,\hat{k}$$

$$\Rightarrow \frac{\partial \vec{R}}{\partial x} = \hat{i} + \frac{\partial f}{\partial x}\hat{k}, \quad \frac{\partial \vec{R}}{\partial y} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

$$\Rightarrow d\vec{S} = \frac{\partial \vec{R}}{\partial x} \times \frac{\partial \vec{R}}{\partial y}\,dx\,dy = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \left[-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right] dx\,dy$$

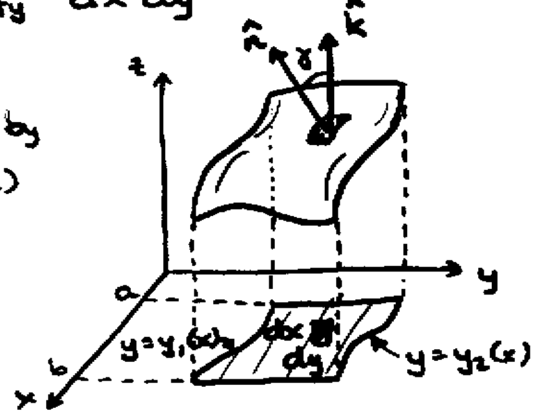
$$\left|\frac{\partial \vec{R}}{\partial x} \times \frac{\partial \vec{R}}{\partial y}\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1 + z_x^2 + z_y^2}$$

$\Rightarrow$ element of surface area (for $z = f(x,y)$)

$$\boxed{dS = \sqrt{1 + \left(\partial f/\partial x\right)^2 + \left(\partial f/\partial y\right)^2}\; dx\,dy}$$

The area of the surface $z = f(x,y)$ over the region D in the x-y plane:

$$S = \iint_D \sqrt{1 + f_x^2 + f_y^2}\; dx\,dy$$



eg if D is the region bounded by $x = a$, $x = b$, $y = y_1(x)$, $y = y_2(x)$

$$\text{Area} = \int_a^b \int_{y_1(x)}^{y_2(x)} \sqrt{1 + f_x^2 + f_y^2}\; dy\,dx$$

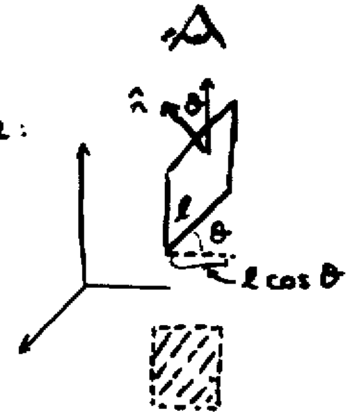
Let γ be the angle between $\hat{k}$ and the outward surface normal $\hat{n}$ (or $d\vec{S} = \hat{n}\,dS$)

Direction cosine $\cos\gamma = \hat{n}\cdot\hat{k} = \dfrac{d\vec{S}\cdot\hat{k}}{dS}$

$$\Rightarrow |\cos\gamma| = |\hat{n}\cdot\hat{k}| = \frac{|d\vec{S}\cdot\hat{k}|}{dS} = \frac{dx\,dy}{\sqrt{1 + f_x^2 + f_y^2}\,dx\,dy} = \left(1 + f_x^2 + f_y^2\right)^{-1/2}$$

$$\Rightarrow \text{area} \quad S = \iint_D \frac{dx\,dy}{|\cos\gamma|} = \iint_D |\sec\gamma|\,dx\,dy$$

We could have also obtained ~~this~~ result using the <u>area cosine principle</u>:



Look at a plane area A, at an angle θ between the line of sight and the normal to the plane; then the apparent area seen is $A\cos\theta$ ← assume θ acute

(distances in one direction appear shorter by a factor $\cos\theta$; distances in the perpendicular direction are unchanged)

Element of surface area dS, normal $\hat{n}$

(dS small – approx. planar

Projection onto x-y plane $dA = dx\,dy$, normal $\hat{k}$

$$\Rightarrow dA = |\cos\gamma|\,dS = |\hat{n}\cdot\hat{k}|\,dS$$

$$\Rightarrow dS = \frac{dA}{|\cos\gamma|} = \frac{\overset{\leftarrow dx\,dy}{dA}}{|\hat{n}\cdot\hat{k}|}$$ as before: surface area $S = \iint_D \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

For a surface given by a graph $z = f(x,y)$:

Normal to $g(x,y,z) = z - f(x,y) = 0$ is $\nabla g = -\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k}$

Unit normal $\hat{n} = \dfrac{-f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \hat{k}}{\sqrt{1+f_x^2+f_y^2}}$

$$\Rightarrow \hat{n}\cdot\hat{k} = \cos\gamma = (1+f_x^2+f_y^2)^{-1/2}$$, as before.

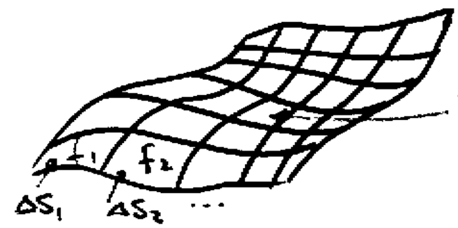
Note: the unit normal vector $\hat{n}$ is given in terms of its direction cosines as

$$\hat{n} = \underbrace{(\cos\alpha)}_{\hat{n}\cdot\hat{\imath}}\,\hat{\imath} + \underbrace{(\cos\beta)}_{\hat{n}\cdot\hat{\jmath}}\,\hat{\jmath} + \underbrace{(\cos\gamma)}_{\hat{n}\cdot\hat{k}}\,\hat{k}$$

Surface integrals

S - a smooth surface
$f(x,y,z)$ defined and continuous on S

Define the surface integral of a scalar field $\int_S f\,dS$ by the usual construction:



(heuristic argument): Subdivide the surface into n pieces, with areas $\Delta S_1, \Delta S_2, \dots, \Delta S_n$ $\quad S = \Delta S_1 \cup \dots \cup \Delta S_n$

In the i^{th} piece, choose a point (x_i, y_i, z_i), and form the sum

$$\sum_{i=1}^{n} \underbrace{f(x_i, y_i, z_i)}_{f_i} \Delta S_i \,. \qquad \text{Now let } n \to \infty \text{ so } \max_i \Delta S_i \to 0$$

- if the limit exists $\quad$ (okay if f is continuous, S a smooth surface)

$\Rightarrow$ Surface integral of f over S:

S not necessarily orientable $\rightarrow$

$$\boxed{\iint_S f\,dS} = \iint_S f(x,y,z)\,dS = \lim_{\substack{n\to\infty \\ \max \Delta S_i \to 0}} \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta S_i$$

If the surface is piecewise smooth, integrate over each smooth part separately, and sum the resulting integrals.

Most commonly f arises from a scalar product:

vector field $\vec{F}$, unit normal $\hat{n} = \dfrac{d\vec{S}}{dS}$ to surface element

(orientable surface) $\Rightarrow f = \vec{F}\cdot\hat{n}$. $\qquad$ $\hat{n}$: unit normal in direction $d\vec{S}$

Total flux of vector field $\vec{F}$ through surface S:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\,dS}$$

Notation: if S is a closed surface, sometimes use $\oiint_S \vec{F}\cdot\hat{n}\,dS$

Recall flux density $\vec{F}$: $|\vec{F}|$: flow rate per unit area per unit time across a surface element normal to $\vec{F}$

$\vec{F}\cdot d\vec{S} = \vec{F}\cdot\hat{n}\,dS$: flux of vector field $\vec{F}$ through element of area dS, outward normal $\hat{n}$

eg $\vec{v} = \vec{v}(x,y,z)$: velocity field of a fluid

$\rho = \rho(x,y,z)$: mass density $\Rightarrow$ mass flux density $\vec{F} = \rho\vec{v}$

$\Rightarrow$ mass of fluid flowing through a small area ΔS with unit normal $\hat{n}$, per unit time, is $\approx \rho\vec{v}\cdot\hat{n}\,\Delta S = \vec{F}\cdot\hat{n}\,\Delta S$

$\Rightarrow$ Rate of flow of fluid across surface S (mass per unit time) is

$$\iint_S \rho\vec{v}\cdot\hat{n}\,dS = \iint_S \rho\vec{v}\cdot d\vec{S}$$

Total flux:

- Parametrized surface $\vec{R}(u,v)$

$$d\vec{S} = \frac{\partial\vec{R}}{\partial u}\times\frac{\partial\vec{R}}{\partial v}\,du\,dv \qquad (u,v)\in D\subset\mathbb{R}^2$$

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\frac{\partial\vec{R}}{\partial u}\times\frac{\partial\vec{R}}{\partial v}\,du\,dv}$$

- Graph of function $z = f(x,y)$

γ: angle between $d\vec{S}$ and $\hat{k}$

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{\underbrace{|\cos\gamma|}_{|\hat{n}\cdot\hat{k}|}}}$$

Also note: surface integrals w.r.t. coordinate elements:

define

$$\iint F_1\,\underbrace{dy\,dz}_{\text{y-z plane: normal }\hat{\imath}} = \iint F_1\,\underbrace{\cos\alpha\,dS}_{\text{projection of } dS \text{ onto y-z plane}} = \iint F_1\,(\hat{\imath}\cdot\hat{n})\,dS$$

$$= \iint_D F_1(\vec{R}(u,v))\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)du\,dv$$

Similarly $\iint F_2\,dz\,dx = \iint F_2\,\underbrace{\cos\beta}_{\hat{\jmath}\cdot\hat{n}}\,dS$, $\iint F_3\,dx\,dy = \iint F_3\,\underbrace{\cos\gamma}_{\hat{k}\cdot\hat{n}}\,dS$

and we have

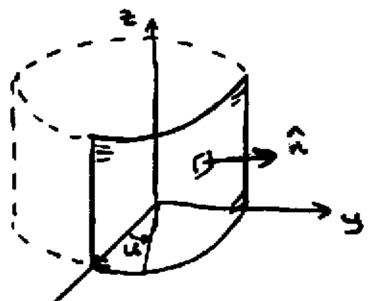
$$\iint \vec{F}\cdot\hat{n}\,dS = \iint (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})\cdot(\cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k})\,dS$$

$$= \iint F_1\,dy\,dz + \iint F_2\,dz\,dx + \iint F_3\,dx\,dy$$

eg compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = z\hat{\imath} + x\hat{\jmath} - 3y^2 z\,\hat{k}$

and S is the surface of the cylinder $x^2+y^2=16$

between $z=0$ and $z=5$, $x>0$, $y>0$: $\hat{n}$: outward normal



$u=\theta,\ v=z$

Method 1: Parametrize surface

$$x = 4\cos u,\quad y = 4\sin u,\quad z = v$$

θ, θ, z — θ, z: cylindrical coordinates

$$0 \le u \le \pi/2, \quad 0 \le v \le 5$$

$$\vec{R}(u,v) = 4\cos u\,\hat{\imath} + 4\sin u\,\hat{\jmath} + v\,\hat{k}$$

$$\Rightarrow \frac{\partial \vec{R}}{\partial u} = -4\sin u\,\hat{\imath} + 4\cos u\,\hat{\jmath},\quad \frac{\partial \vec{R}}{\partial v} = \hat{k}\ ;\quad \vec{F} = v\,\hat{\imath} + 4\cos u\,\hat{\jmath} - 48\sin^2 u\cdot v\,\hat{k}$$

$$\Rightarrow \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} = 4\sin u\,\hat{\jmath} + 4\cos u\,\hat{\imath}$$

$$\Rightarrow \vec{F}\cdot\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} = \left|\begin{pmatrix} v & 4\cos u & -48\sin^2 u\cdot v \\ -4\sin u & 4\cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 4v\cos u + 16\cos u\sin u$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \int_0^5\int_0^{\pi/2} \vec{F}\cdot\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\,du\,dv = \int_0^5\int_0^{\pi/2}(4v\cos u + 8\sin 2u)\,du\,dv$$

$$= \int_0^5 \Big[\,4v\sin u - 4\cos 2u\,\Big]\Big|_0^{\pi/2} dv = \int_0^5 (4v+8)\,dv$$

$$= (2v^2+8v)\Big|_0^5 = 90.$$

Method 2: Project onto coordinate plane, use area cosine principle :

Normal to surface $\underbrace{x^2+y^2}_{g(x,y,z)}=16$ is $2x\hat{\imath} + 2y\hat{\jmath}$,

unit normal $\dfrac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{\underbrace{x^2+y^2}_{16}}} = \dfrac{x}{4}\hat{\imath} + \dfrac{y}{4}\hat{\jmath}$

Surface is $y = \sqrt{16-x^2}$, $0 \le x \le 4$, $0 \le z \le 5$ - use x, z as parameters

- then $\vec{F} = z\hat{\imath} + x\hat{\jmath} - 3(16-x^2)z\,\hat{k}$

$$\Rightarrow \vec{F}\cdot\hat{n} = \frac{xz}{4} + \frac{xy}{4}$$

ie we wish to compute $\iint_S \frac{x}{4}(y+z)\,dS$ with $y = \sqrt{16-x^2}$

- integrate over x-z plane, unit normal $\hat{\jmath}$, direction cosine

$$\cos\beta = \hat{\jmath}\cdot\hat{n} = y/4 \Rightarrow dS = \frac{dx\,dz}{\cos\beta}$$

Thus

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \frac{x}{4}(y+z)\frac{dx\,dz}{\hat{\jmath}\cdot\hat{n}} = \iint \frac{x}{4}(y+z)\frac{4}{y}\,dx\,dz$$

$$= \int_0^5\int_0^4 x\left(1+\frac{z}{\sqrt{16-x^2}}\right)dx\,dz = \int_0^5 \left[\tfrac{1}{2}x^2 - z\sqrt{16-x^2}\right]_0^4 dz$$

$$= \int_0^5 (8+4z)\,dz = (8z+2z^2)\Big|_0^5 = 90$$, as before.

---

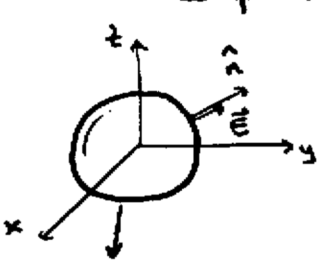
eg $\vec{E} = -\nabla\left(\frac{1}{|\vec{R}|}\right) = \frac{\vec{R}}{|\vec{R}|^3} = \frac{\hat{e}_r}{r^2}$ $\qquad \vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, $|\vec{R}| = \sqrt{x^2+y^2+z^2}$

(recall: in spherical coordinates, $f = -\frac{1}{r}$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial r}\hat{e}_r = \frac{1}{r^2}\hat{e}_r = \frac{1}{|\vec{R}|^2}\frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{|\vec{R}|^3}\text{)}$$

Compute $\iint_S \vec{E}\cdot d\vec{S}$, where S is the surface of the sphere

$$|\vec{R}|^2 = x^2+y^2+z^2 = 9 \quad = \underbrace{a^2}_{\text{radius}}$$



On S, the outward normal direction is radial (direction $\vec{R}$)

$$\Rightarrow \hat{n} = \hat{e}_r = \frac{\vec{R}}{|\vec{R}|} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \tfrac{1}{3}(x\hat{\imath}+y\hat{\jmath}+z\hat{k}) = \frac{\vec{R}}{3}$$

$a \to 3$

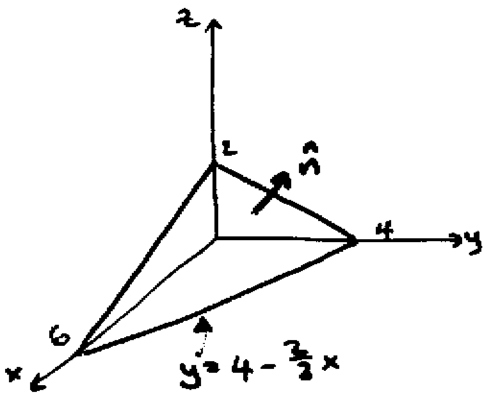
$$\Rightarrow \vec{E}\cdot\hat{n} = \frac{\vec{R}\cdot\vec{R}}{|\vec{R}|^4} = \frac{1}{|\vec{R}|^2} = \frac{1}{9} = \text{constant}$$

$\approx 1/a^2$ $\qquad$ ($|\vec{E}|$ is constant on S by symmetry)

$$\Rightarrow \iint_S \vec{E}\cdot\hat{n}\,dS = \frac{1}{9}\iint_S dS = \underbrace{\frac{1}{9}}_{1/a^2}\left(\begin{array}{l}\text{total surface area}\\ \text{of sphere, radius } 3\end{array}\right) = \underbrace{\frac{1}{9}}_{1/a^2}\cdot 4\pi\cdot\underbrace{3^2}_{a^2} = 4\pi$$

independent of radius

eg compute the surface integral of the normal component of
$$\vec{F} = 18xz\,\hat{\imath} + 6xy\,\hat{\jmath} + (2x^2+3)\,\hat{k}$$
over the triangle with vertices $(6,0,0)$, $(0,4,0)$, $(0,0,2)$
(oriented so that the positive side is away from the origin).



The surface of the triangle lies in the plane
$$2x+3y+6z=12, \qquad \leftarrow \text{on } z=0,\ y = 4-\tfrac{2}{3}x$$
with unit normal $\hat{n} = \dfrac{2\hat{\imath}+3\hat{\jmath}+6\hat{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}$

Hence $\vec{F}\cdot\hat{n} = \frac{36}{7}xz + \frac{18}{7}xy + \frac{6(2x^2+3)}{7}$

We can parametrize the surface by $z = 2 - \frac{1}{3}x - \frac{1}{2}y = f(x,y)$,
so that $dS = \sqrt{1+f_x^2+f_y^2}\,\underbrace{dx\,dy}_{dA} = \sqrt{1+\frac{1}{9}+\frac{1}{4}}\,dx\,dy = \sqrt{\frac{49}{36}}\,dx\,dy$
$= \frac{7}{6}\,dx\,dy$

[or: $\vec{R} = x\hat{\imath} + y\hat{\jmath} + (2-\frac{1}{3}x - \frac{1}{2}y)\hat{k} \Rightarrow \frac{\partial\vec{R}}{\partial x} = \hat{\imath} - \frac{1}{3}\hat{k}$, $\frac{\partial \vec{R}}{\partial y} = \hat{\jmath} - \frac{1}{2}\hat{k}$
$\Rightarrow \frac{\partial\vec{R}}{\partial x}\times\frac{\partial\vec{R}}{\partial y} = \hat{k} + \frac{1}{2}\hat{\jmath} + \frac{1}{3}\hat{\imath}$, $dS = \left|\frac{\partial\vec{R}}{\partial x}\times\frac{\partial\vec{R}}{\partial y}\right| dx\,dy = \frac{7}{6}dx\,dy$

or: $\underset{\cos\gamma}{\hat{n}\cdot\hat{k}} = \left(\frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}\right)\cdot\hat{k} = \frac{6}{7} \Rightarrow dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{7}{6}\,dx\,dy$

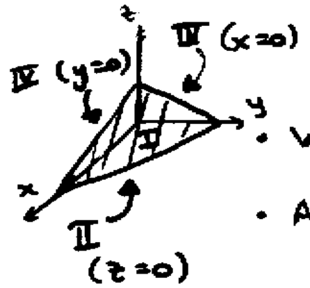
Of course we could also parametrize the surface over the x-z or y-z planes.]

Thus $\vec{F}\cdot\hat{n}\,dS = \vec{F}\cdot\hat{n}\,\frac{7}{6}\,dx\,dy = (6xz + 3xy + 2x^2+3)\,dx\,dy$,

and on the surface S: $z = 2 - \frac{1}{3}x - \frac{1}{2}y$, we have
$$\vec{F}\cdot\hat{n}\,dS\big|_S = [6x(2-\tfrac{1}{3}x-\tfrac{1}{2}y) + 3xy + 2x^2+3]\,dx\,dy = (12x+3)\,dx\,dy$$

Thus $$\iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^6\int_0^{4-\frac{2}{3}x}(12x+3)\,dy\,dx = \int_0^6 (12x+3)(4-\tfrac{2}{3}x)\,dx$$
$$= \int_0^6 (12+46x-8x^2)\,dx = \left(12x+23x^2-\tfrac{8}{3}x^3\right)\Big|_0^6$$
$$= 324.$$

eg compute $\iint \vec{F}\cdot d\vec{S}$ over the surface of the tetrahedron with vertices $(0,0,0)$, $(6,0,0)$, $(0,4,0)$, $(0,0,2)$, and $\vec{F} = 18xz\,\hat{\imath} + 6xy\,\hat{\jmath} + (2x^2+3)\hat{k}$



- We have already computed the integral over one (slanted) surface, I.
- Along the bottom face, the outward normal is $\hat{n} = -\hat{k}$, and

  $dS = dx\,dy$, so $\vec{F}\cdot\hat{n}\,dS = -\vec{F}\cdot\hat{k}\,dx\,dy = -(2x^2+3)\,dx\,dy$;

  this face lies in the x-y plane $z=0$.

  Thus the surface integral is

$$\iint_{S_{II}} \vec{F}\cdot\hat{n}\,dS = -\int_0^6 \int_0^{4-\frac{2}{3}x} (2x^2+3)\,dy\,dx = -\int_0^6 (2x^2+3)(4-\tfrac{2}{3}x)\,dx$$

$$= -\int_0^6 \left(12 - 2x + 8x^2 - \tfrac{4}{3}x^3\right)dx = \left(-12x + x^2 - \tfrac{8}{3}x^3 + \tfrac{1}{3}x^4\right)\Big|_0^6$$

$$= -180$$

- The back face III in the y-z plane has outward normal $\hat{n} = -\hat{\imath}$,

  so $\vec{F}\cdot\hat{n}\,dS = -18xz\,dy\,dz$; but on this face $\underline{x=0}$, so

$$\iint_{S_{III}} \vec{F}\cdot\hat{n}\,dS = \iint 0\,dy\,dz = 0$$

- Similarly, on the left face IV in the x-z plane, $\hat{n} = -\hat{\jmath}$, so $\vec{F}\cdot\hat{n} = -6xy$

  but on this face $\underline{y=0}$, so $\iint_{S_{IV}} \vec{F}\cdot\hat{n}\,dS = \iint 0\,dx\,dz = 0$.

- In summary, the net flux through the surface is

$$\iint_S \vec{F}\cdot\hat{n}\,dS = 324 - 180 + 0 + 0 = 144.$$

Applications

- eg heat flow $\quad T(x,y,z)$: temperature field (or $T(\vec{R},t)$)

  $\vec{F}(x,y,z)$: heat flux density (often $\vec{Q} = \vec{Q}(\vec{R},t)$, or $\vec{q}$)

units eg $\frac{\text{calories}}{\text{time}\cdot\text{area}} \rightarrow$ $|\vec{F}|$: rate of heat flow per unit area across a surface element perpendicular to $\vec{F}$

$\vec{F}\cdot\hat{n}$: heat flow per unit area, per unit time across a surface element with normal $\hat{n}$

Fourier's Law $\vec{F} = -k\nabla T$: heat flux is proportional to the temperature gradient

(k: thermal conductivity)

$\Rightarrow$ Total rate of heat flow out of a region bounded by a surface S is $\iint_S \vec{F}\cdot\hat{n}\,dS = -\iint_S k\nabla T\cdot\hat{n}\,dS$

· eg electrostatics : electric field $\vec{E}$ $\leftarrow$ force on a charge q_1 in an electric field $\vec{E}$ is $\vec{F} = q_1\vec{E}$.

Coulomb's Law $\vec{E} = kq\dfrac{\vec{R}}{|\vec{R}|^3} = kq\dfrac{\hat{e}_r}{r^2} = -\nabla\phi$,

(electric field at $\vec{R}$ due to a point charge q at the origin)

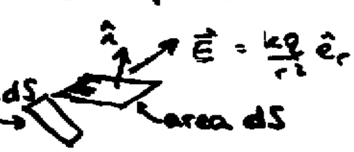
$\phi = -\dfrac{kq}{|\vec{R}|}$ (ϕ: electrostatic potential) $\qquad k = \dfrac{1}{4\pi\varepsilon_0}$ (SI units)

$\Rightarrow$ flux of electric field over a sphere, radius a, with centre at the charge q is

$$\iint_{S_a}\vec{E}\cdot\hat{n}\,dS = \frac{q}{4\pi\varepsilon_0}\iint_{S_a}\frac{\vec{R}}{|\vec{R}|^3}\cdot\hat{n}\,dS = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{a^2}\cdot 4\pi a^2 = \frac{q}{\varepsilon_0}$$

$\hat{n} = \dfrac{\vec{R}}{|\vec{R}|} = \hat{e}_r$ (independent of a)

– in fact the flux is also independent of the shape of the surface enclosing the charge q



$\vec{E}\cdot\hat{n}\,dS$ depends only on the solid angle $d\Omega = \sin\phi\,d\phi\,d\theta$ subtended by the surface element dS, and is independent of the surface normal $\hat{n}$ or radius r

· electric fields are additive (if charge q_1 induces a field $\vec{E}_1$ at $\vec{R}$, and the field due to q_2 is $\vec{E}_2$, then the field due to q_1 and q_2 is $\vec{E}_1 + \vec{E}_2$).

$\rightarrow$ Gauss' Law : Electric flux across S $\iint_S \vec{E}\cdot\hat{n}\,dS = \dfrac{q}{\varepsilon_0}$

(S any closed surface)

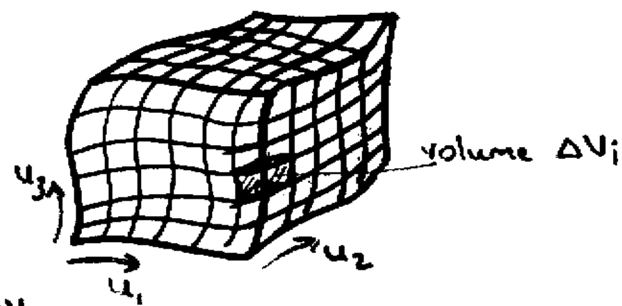
where q : total charge enclosed by S.

Volume integrals

f - a scalar field, defined and continuous in a bounded domain $V \subset \mathbb{R}^3$

V bounded ie V is contained in a sufficiently large cube R: $x_1 \le x \le x_2$, $y_1 \le y \le y_2$, $z_1 \le z \le z_2$.

Define the volume integral of a scalar field f, $\int_V f \, dV$ in the usual way:



Partition the domain V into n regions with volumes $\Delta V_1, \Delta V_2, \dots, \Delta V_n$

$V = \Delta V_1 \cup \dots \cup \Delta V_n$

In the i^{th} region, select a point P_i with position vector $\vec{R}_i$, coordinates (u_i, v_i, w_i) in some coordinate system, at which f takes the value $f_i = f(\vec{R}_i)$. Form the sum $\sum_{i=1}^{n} f_i \, \Delta V_i$. Now let $n \to \infty$ so that $\max_i \Delta V_i \to 0$.

$= f(u_i, v_i, w_i)$

If the limit exists (okay if f is continuous, boundary ∂V of V smooth)

$\Rightarrow$ volume integral of f over V:

$$\boxed{\iiint_V f \, dV} = \lim_{\substack{n \to \infty \\ \max_i \Delta V_i \to 0}} \sum_{i=1}^{n} \underbrace{f(u_i, v_i, w_i)}_{f_i} \Delta V_i$$

Notation: Sometimes $d\vec{x}$ or d^3x (in $\mathbb{R}^3$)

(For a vector field $\vec{F} = F_1 \hat{\imath} + F_2 \hat{\jmath} + F_3 \hat{k}$, $\iiint_V \vec{F} \, dV = \hat{\imath} \iiint_V F_1 \, dV + \hat{\jmath} \iiint_V F_2 \, dV + \hat{k} \iiint_V F_3 \, dV$

eg in Cartesian coordinates, $dV = dx \, dy \, dz$

$$\Rightarrow \iiint_V f \, dV = \iiint_V f(x,y,z) \, dx \, dy \, dz$$

eg in orthogonal curvilinear coordinates (u_1, u_2, u_3), with $h_i = \left| \frac{\partial \vec{R}}{\partial u_i} \right|$

$$dV = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3$$

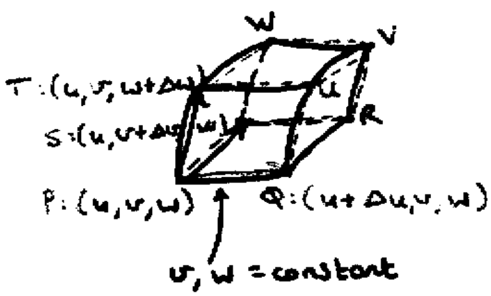
$$\Rightarrow \iiint_V f \, dV = \iiint_V f(u_1, u_2, u_3) \, dV = \iiint_V f(u_1, u_2, u_3) \, h_1 h_2 h_3 \, du_1 \, du_2 \, du_3$$

eg cylindrical polar coordinates: $dV = \rho\, d\rho\, d\theta\, dz$

spherical polar coordinates: $dV = r^2 \sin\phi\, dr\, d\phi\, d\theta$

For a general coordinate system in $\mathbb{R}^3$, $\left.\begin{array}{l} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{array}\right\}$

Regular volume element:



Consider a small region bounded by surfaces of constant u, v, and w, with volume ΔV.

For small $\Delta u, \Delta v, \Delta w$, we approximate the region by a parallelepiped:

$$\vec{PQ} \approx \vec{R}(u+\Delta u, v, w) - \vec{R}(u,v,w) \approx \frac{\partial \vec{R}}{\partial u}\Delta u$$

Similarly $\vec{PS} \approx \dfrac{\partial \vec{R}}{\partial v}\Delta v$, $\vec{PT} \approx \dfrac{\partial \vec{R}}{\partial w}\Delta w$

Volume of parallelepiped is given by absolute value of scalar triple product

$$\text{Volume } \Delta V \approx |\vec{PQ}\cdot\vec{PS}\times\vec{PT}| \approx \left|\left[\frac{\partial \vec{R}}{\partial u}, \frac{\partial \vec{R}}{\partial v}, \frac{\partial \vec{R}}{\partial w}\right]\right| \Delta u\, \Delta v\, \Delta w$$

Now $\vec{R}(u,v,w) = x(u,v,w)\,\hat{\imath} + y(u,v,w)\,\hat{\jmath} + z(u,v,w)\,\hat{k}$

$$\Rightarrow \frac{\partial \vec{R}}{\partial u} = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k} \quad \left(\text{similarly for } \frac{\partial \vec{R}}{\partial v}, \frac{\partial \vec{R}}{\partial w}\right)$$

and in terms of components, the triple product is given by the (Jacobian) determinant

$$\frac{\partial \vec{R}}{\partial u}\cdot\frac{\partial \vec{R}}{\partial v}\times\frac{\partial \vec{R}}{\partial w} = \det \underbrace{\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}}_{\text{Jacobian matrix}} = \det \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

using $\det J = \det J^T$

$$\Rightarrow \text{volume element } dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$

<u>Applications</u> • eg $\rho(\vec{R}) = \rho(x,y,z)$: <u>mass density</u> of a material

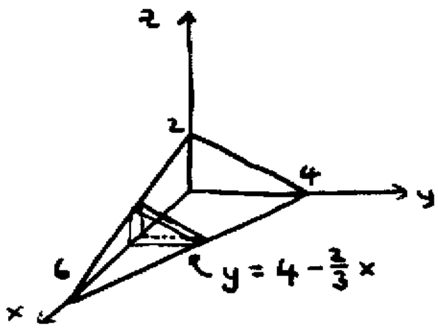
$\Rightarrow$ <u>total mass</u> of domain V: $M = \iiint_V \rho\, dV$

- eg $\rho(\vec{R})$: <u>charge density</u>

$\Rightarrow$ <u>total charge</u> contained in V is $Q = \iiint_V \rho\, dV$

Typically, volume integrals are evaluated by iterated integration

eg compute the volume integral of $f = 6x + 18z$ over the tetrahedron with vertices at $(0,0,0)$, $(6,0,0)$, $(0,4,0)$, $(0,0,2)$ (ie over the region bounded by the planes $x=0$, $y=0$, $z=0$, and $2x+3y+6z=12$)



$$\iiint_V f\,dV = \int_0^6 \int_0^{4-\frac{2}{3}x} \int_0^{2-\frac{1}{3}x-\frac{1}{2}y} (6x+18z)\,dz\,dy\,dx$$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} \left(6xz + 9z^2\right)\Big|_0^{2-\frac{1}{3}x-\frac{1}{2}y} dy\,dx$$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} \left(36 - x^2 - 18y + \frac{9}{4}y^2\right) dy\,dx$$

$$= \int_0^6 \left(36y - x^2y - 9y^2 + \frac{3}{4}y^3\right)\Big|_0^{4-\frac{2}{3}x} dx = \int_0^6 \left(48 - 4x^2 + \frac{4}{9}x^3\right) dx$$

$$= \left(48x - \frac{4}{3}x^3 + \frac{1}{9}x^4\right)\Big|_0^6 = 144$$

Note that for $\vec{F} = 18xz\,\hat{\imath} + 6xy\,\hat{\jmath} + (2x^2+3)\,\hat{k}$, we have $\nabla\cdot\vec{F} = 18z + 6x = f$,

so we have shown for this vector field $\vec{F}$ and this tetrahedron, that $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$, where $S = \partial V$ is the surface enclosing the domain V

- a special case of the <u>divergence theorem</u>.

---

eg compute the volume integral of $f(x,y,z) = x$ over the ball $x^2+y^2+z^2 \le a^2$ (interior of the sphere $x^2+y^2+z^2 = a^2$) : (expect 0 by symmetry)

Use <u>spherical polar</u> coordinates : $x = r\sin\phi\cos\theta$, $dV = r^2\sin\phi\,dr\,d\phi\,d\theta$

$$\Rightarrow \iiint_V f\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (r\sin\phi\cos\theta)\,r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$= 0 \quad , \quad \text{since } \int_0^{2\pi}\cos\theta\,d\theta = 0.$$

Integral Theorems of Vector Analysis

$\vec{F}$: a (piecewise) smooth vector field

- **Green's Theorem in the Plane**

  D - a domain in the x-y plane, bounded by a regular closed curve $C = \partial D$, oriented counterclockwise:

$$\boxed{\oint_C (F_1\,dx + F_2\,dy) = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy}$$

- (Gauss) **Divergence Theorem**

  V - a domain in $\mathbb{R}^3$ bounded by a regular surface $S = \partial V$:
  closed

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\,dV}$$

(the total divergence of $\vec{F}$ in V equals the net flux of $\vec{F}$ through the surface S of V)

- **Stokes' Theorem**

  S - a surface in $\mathbb{R}^3$ bounded by a regular closed curve $C = \partial S$ (where the orientation of C is positive in terms of the orientation of S given by $\hat{n}$)

$$\boxed{\oint_C \vec{F}\cdot d\vec{R} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}}$$

(the total flux of the curl of a vector field $\vec{F}$ through a surface S equals the line integral of $\vec{F}$ around the edge).

Note: the divergence theorem applies to closed surfaces, with no boundary curve; Stokes' theorem applies to open surfaces bounded by a regular curve C.

Divergence Theorem

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_{S} \vec{F}\cdot\hat{n}\,dS$$

S ← ∂V

V: a domain bounded by surface S

Heuristic "proof":

Divergence $\operatorname{div}\vec{F}$ is "net flux per unit volume"

$\Rightarrow$ net flux out through the surface of a small volume ΔV is $(\nabla\cdot\vec{F})\Delta V = (\nabla\cdot\vec{F})\,\Delta x \Delta y \Delta z$ (for a rectangular block: parallelepiped)

$\iint_{S_i} \vec{F}\cdot\hat{n}\,dS$

Consider a general domain V: divide it into many small rectangular blocks, and sum the flux out of all the blocks, to get

$$\sum_{i=1}^{n} (\nabla\cdot\vec{F})_i\,\Delta V_i \approx \iiint_V (\nabla\cdot\vec{F})\,dV$$ ← sum approaches volume integral over V

For faces not on the surface S of V, the flux outward from one block equals the inward flux over the common face of an adjacent block, and in the sum their contributions cancel

$\Rightarrow$ in $\sum_{i=1}^{n} \iint_{S_i} \vec{F}\cdot\hat{n}\,dS$, the only terms that remain in the sum of the fluxes are the fluxes out of faces at the surface S, so the sum gives $\iint_S \vec{F}\cdot\hat{n}\,dS$.

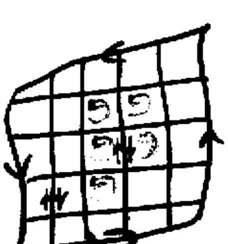
Stokes' Theorem

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{R}$$

S: a surface bounded by a curve C

Heuristic "proof":

Curl $\operatorname{curl}\vec{F}$ is "circulation per unit area"



$\Rightarrow$ for a small rectangle of area ΔS, normal $\hat{n}$, $(\nabla\times\vec{F})\cdot\hat{n}\,\Delta S \approx \oint \vec{F}\cdot\underbrace{\vec{T}\,ds}_{d\vec{R}}$

In computing $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$, we are effectively summing all the line integrals over many small rectangles subdividing the surface S; but the line integrals cancel over internal edges (since they are counted once in each direction), which leaves the line integral around the boundary curve C.

Green's Theorem (in the Plane) – a two-dimensional result.

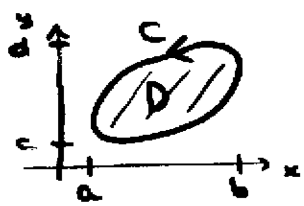
Theorem:

Let D be a domain in the x-y plane bounded by a regular closed curve C, oriented counterclockwise (ie. choosing $+\hat{k}$ as the unit normal to the plane).

Let $F_1(x,y)$ and $F_2(x,y)$ be continuous functions of x and y, and let $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ exist and be continuous throughout the domain D. (sufficient: F_1, F_2 have continuous partial derivatives on D)

Then
$$\oint_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \underbrace{dx\,dy}_{dA}$$

Proof: First assume that D has the property that any line passing through an interior point of D and parallel to either coordinate axis, cuts the boundary $C = \partial D$ in exactly two places

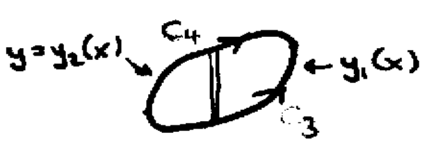


ie D can be represented in both the forms
$$a \le x \le b, \quad y_1(x) \le y \le y_2(x)$$
$$\text{and} \quad c \le y \le d, \quad x_1(y) \le x \le x_2(y)$$

Then
$$\iint_D \frac{\partial F_2}{\partial x}\,dA = \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial F_2}{\partial x}\,dx\,dy$$
(iterated integral: integrate first wrt x, then y)

$C = C_2 - C_1$

$$= \int_c^d \left[F_2(x_2(y),y) - F_2(x_1(y),y)\right] dy$$
$$= \int_{C_2} F_2\,dy - \int_{C_1} F_2\,dy = \oint_C F_2\,dy$$



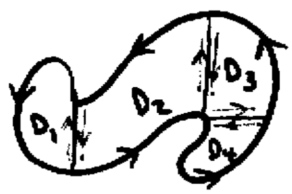
$C = C_3 - C_4$

and similarly
$$-\iint_D \frac{\partial F_1}{\partial y}\,dA = -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial F_1}{\partial y}\,dy\,dx$$
(integrate first w.r.t. y, then x)

$$= \underbrace{\int_a^b F_1(x, y_1(x))\,dx}_{\int_{C_3} F_1\,dx} - \underbrace{\int_a^b F_1(x, y_2(x))\,dx}_{\int_{C_4} F_1\,dx} = \oint_C F_1\,dx$$

as required.

Next, if D is a region that can be decomposed into finitely many regions $D_1, D_2, \dots, D_n$ with boundaries $C_1, C_2, \dots, C_n$ having the above properties, we sum the integrals involved over all regions.



From above,

$$\oint_{C_1}(F_1\,dx + F_2\,dy) + \dots + \oint_{C_n}(F_1\,dx + F_2\,dy)$$
$$= \iint_{D_1}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy + \dots + \iint_{D_n}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

The double integrals cover all the (disjoint) subdomains, and thus just add to $\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$

If two regions D_i, D_j are adjacent, the line integrals will be taken over their common arc once in each direction. In the sum over the line integrals, the contributions along the added internal arcs thus cancel pairwise, while the remaining terms add up to precisely the integral around the boundary C of D, taken counterclockwise: $\oint_C (F_1\,dx + F_2\,dy)$

The proof so far covers all domains D of practical interest.
Finally for the most general simply connected domain D, one may approximate the domain by regions of the special type considered above, and use a limiting process.

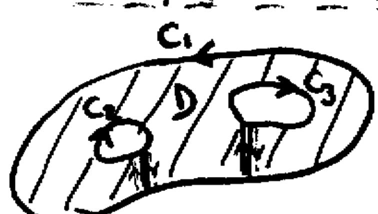
$\boxtimes$

Application: area

By Green's theorem, the area of the domain D is

$$\text{area of } D = \iint_D 1\,dx\,dy = \oint_C x\,dy = \oint_C (-y)\,dx = \tfrac{1}{2}\oint_C(-y\,dx + x\,dy)$$

Multiply connected regions:



$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \left[\oint_{C_1} + \oint_{C_2} + \oint_{C_3}\right](F_1\,dx + F_2\,dy)$$

Recall: the positive direction on the boundary curve C keeps the region D on the left.

Vector forms of Green's Theorem:

$$\vec{F}(x,y) = F_1(x,y)\,\hat{\imath} + F_2(x,y)\,\hat{\jmath}$$ vector field in the plane

- D: surface in x-y plane, normal $\hat{n} = \hat{k}$

$d\vec{S} = \hat{k}\,dx\,dy = \hat{k}\,dA$; along boundary curve C, $d\vec{R} = dx\,\hat{\imath} + dy\,\hat{\jmath}$

$\nabla\times\vec{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$ $\Rightarrow \vec{F}\cdot d\vec{R} = F_1\,dx + F_2\,dy$

Green's Theorem is

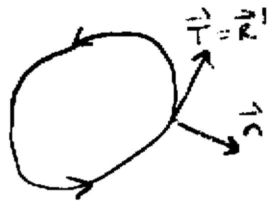
$$\underbrace{\oint_C F_1\,dx + F_2\,dy}_{} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{R} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\cdot\underbrace{\hat{k}\,dx\,dy}_{d\vec{S}}$$

$$= \iint_D (\nabla\times\vec{F})\cdot d\vec{S}$$

ie Green's theorem is a special case of Stokes' theorem in two dimensions

- Parametrize the curve C: $\vec{R}(s) = x(s)\,\hat{\imath} + y(s)\,\hat{\jmath}$



$\Rightarrow$ unit tangent $\vec{T}(s) = \frac{d\vec{R}}{ds} = \frac{dx}{ds}\,\hat{\imath} + \frac{dy}{ds}\,\hat{\jmath}$

$\Rightarrow$ outward normal $\vec{n}(s) = \vec{T}\times\hat{k} = \frac{dy}{ds}\,\hat{\imath} - \frac{dx}{ds}\,\hat{\jmath}$

Then

$$\underbrace{\oint_C \vec{F}\cdot\hat{n}\,ds}_{\text{flux through curve C}} = \int_0^L \left(F_1\frac{dy}{ds} - F_2\frac{dx}{ds}\right)ds = \oint_C (-F_2\,dx + F_1\,dy)$$

$$\overset{\text{Green's theorem for } \vec{C}(x,y) = -F_2\,\hat{\imath} + F_1\,\hat{\jmath}}{=} \iint_D \left(\frac{\partial F_1}{\partial x} - \frac{\partial}{\partial y}(-F_2)\right)dx\,dy = \iint_D \nabla\cdot\vec{F}\,dx\,dy$$

ie $$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \nabla\cdot\vec{F}\,dx\,dy$$

ie Green's theorem is a special case of the divergence theorem in two dimensions

Examples : Divergence & Stokes Theorem

eg Verify the divergence theorem for $\vec{F} = r^n\,\hat{e}_r$ (spherical coordinates) where V is the upper hemisphere of a ball of radius 3, bounded below by $z=0$, and above by $r=3$, $0 \le \phi \le \pi/2$:

Volume integral:

In spherical coordinates, $\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^n) + 0 + 0$

$= \frac{1}{r^2}(n+2)r^{n+1} = (n+2)r^{n-1}$

$$\Rightarrow \iiint_V \nabla\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (n+2)\,r^{n-1}\,\underbrace{r^2\sin\phi\,dr\,d\phi\,d\theta}_{dV}$$

$$= \cancel{(n+2)}\,2\pi\,(-\cos\phi)\Big|_0^{\pi/2}\frac{1}{\cancel{(n+2)}}r^{n+2}\Big|_0^3 = 2\pi\cdot 3^{n+2}$$

Surface integral:

On upper hemisphere, $\hat{n} = \hat{e}_r \Rightarrow \vec{F}\cdot\hat{n}\,|_S = r^n\big|_{r=3} = 3^n$

Element of area $dS = 3^2\sin\phi\,d\phi\,d\theta$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^{\pi/2} 3^{n+2}\sin\phi\,d\phi\,d\theta = 3^{n+2}\cdot 2\pi$$

On lower surface, $\hat{n} = -\hat{k}$, while on $z=0$,

$\hat{e}_r$ lies in the x-y plane $\left(\vec{F}\big|_{z=0} = (x^2+y^2)^{n-1}(x\hat{i}+y\hat{j})\right)$

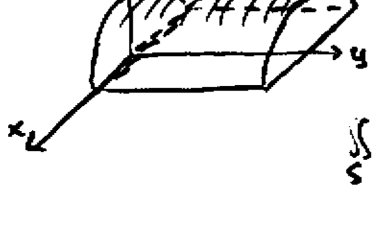
$\Rightarrow \vec{F}\cdot\hat{n}\,\big|_{z=0} = 0$: no further contribution from this surface

thus $\iint_{\partial V}\vec{F}\cdot d\vec{S} = 2\pi\cdot 3^{n+2} = \iiint_V \nabla\cdot\vec{F}\,dV.$

---

eg Compute the outward flux of $\vec{F} = (x+\cos y)\,\hat{i} + (y+\sin(x-z^3))\,\hat{j} + (z+e^{x-y})\,\hat{k}$

through the surface of the region V bounded by the planes $z=0$, $y=0$, $y=2$, and the parabolic cylinder $z = 1-x^2$:



Computing $\iint_S \vec{F}\cdot\hat{n}\,dS$ directly would be difficult...

But $\nabla\cdot\vec{F} = 3 \Rightarrow$ by the divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV = 3\iiint_V dV = 3\int_{-1}^{1}\int_0^2\int_0^{1-x^2} dz\,dy\,dx$$

$$= 3\cdot 2\cdot\int_{-1}^{1}(1-x^2)\,dx = 6\cdot\frac{4}{3} = 8.$$

Note: $\vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \Rightarrow \nabla\cdot\vec{R} = 3$,

so the volume of a region may be computed via a surface integral by
$$\text{vol}(V) = \iiint_V dV = \frac{1}{3}\iiint_V \nabla\cdot\vec{R}\, dV = \frac{1}{3}\iint_S \vec{R}\cdot\hat{n}\, dS$$

eg verify Stokes' Theorem for $\vec{F} = (2x-y)\hat{\imath} - yz^2\hat{\jmath} - y^2 z\hat{k}$ over the upper hemispherical surface of radius 1 : $r=1,\ 0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi$

Line integral:

The boundary curve C is $x^2+y^2=1$; parametrize by $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

$\Rightarrow \vec{R} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad \Rightarrow \dfrac{d\vec{R}}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$(0 \le \theta \le 2\pi)$

On C, $z=0$, so $\vec{F} = (2x-y)\hat{\imath} = (2\cos\theta - \sin\theta)\,\hat{\imath}$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \vec{F}(\vec{R}(\theta))\cdot\frac{d\vec{R}}{d\theta}\,d\theta = \int_0^{2\pi}(2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$
$$= -\int_0^{2\pi}\sin 2\theta\, d\theta + \int_0^{2\pi}\sin^2\theta\, d\theta = 0 + \tfrac{1}{2}\cdot 2\pi = \pi$$

Surface integral:

$$\nabla\times\vec{F} = (-2yz + 2yz)\hat{\imath} + (0-0)\hat{\jmath} + (0+1)\hat{k} = \hat{k}$$

Outward normal on $r=1$: $\hat{e}_r\big|_{r=1} = \dfrac{\vec{R}}{\underbrace{|\vec{R}|}_{=1}} = \vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\Rightarrow (\nabla\times\vec{F})\cdot\hat{n}\big|_{r=1} = \hat{k}\cdot\vec{R} = z = \cos\phi$$

Parametrize surface of sphere by angles ϕ, θ:

$dS = 1^2\sin\phi\, d\phi\, d\theta$

$$\Rightarrow \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS = \int_0^{2\pi}\int_0^{\pi/2}\cos\phi\,\sin\phi\, d\phi\, d\theta$$
$$= \tfrac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\cdot 2\pi = \tfrac{1}{2}\cdot 2\pi = \pi$$
$$= \oint_C \vec{F}\cdot d\vec{R}$$

We can use Stokes' Theorem to complete the proof of the relationship between irrotational and conservative vector fields on a simply connected domain:

Thm: $\vec{F}$ is irrotational, $\nabla\times\vec{F}=\vec{0}$, on a simply connected domain iff $\vec{F}$ is conservative, $\vec{F}=\nabla\phi$.

Proof: Conservative $\Rightarrow$ irrotational: $\nabla\times\nabla\phi=\vec{0}$.

Irrotational $\Rightarrow$ conservative: By a previous result, it is sufficient to show that the line integral of $\vec{F}$ around any closed curve C vanishes; this is equivalent to $\vec{F}=\nabla\phi$ for some ϕ.

Let S be any surface with boundary C.

By Stokes' Theorem, $\oint_C \vec{F}\cdot d\vec{R} = \iint_S \underbrace{(\nabla\times\vec{F})}_{\substack{=\vec{0} \\ \text{by assumption}}}\cdot d\vec{S} = \iint_S \vec{0}\, d\vec{S} = 0.$ ∎

Applications

Conservation Laws:

ρ: density of some quantity (amount per unit volume)
eg mass density ρ

$\vec{F}$: the corresponding flux density of that quantity (often: $\vec{J}$)
eg mass flux density $\vec{F}=\rho\vec{v}$

Rate of change of total mass in volume V
= rate at which mass flows into V
= $-$ (flux of mass out of V)

More generally:
$\frac{d}{dt}$(total quantity in V)
$= -$ flux out of V

$$\boxed{\frac{d}{dt}\underbrace{\iiint_V \rho\, dV}_{\text{total mass}} = -\iint_{S=\partial V} \vec{F}\cdot\hat{n}\, dS}$$

Integral form of Conservation Law

Now $\frac{d}{dt}\iiint_V \rho(x,y,z,t)\, dV = \iiint_V \frac{\partial\rho}{\partial t}(x,y,z,t)\, dV$ (integral w.r.t. x,y,z)

By the divergence theorem,

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \nabla\cdot\vec{F}\, dV$$ assume $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{F}$ is continuous

$$\Rightarrow \iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{F}\right) dV = 0$$ but this is true for any volume V, so we have

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{F} = 0}$$ Equation of Continuity

[Differential Form of Conservation Law]

eg • mass density ρ, mass flux density $\vec{F} = \rho\vec{v}$:

Conservation of Mass : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$

Note : $\nabla\cdot(\rho\vec{v}) = \rho\nabla\cdot\vec{v} + \vec{v}\cdot\nabla\rho$

ρ = constant, mass conservation $\Rightarrow \nabla\cdot\vec{v} = 0$

• heat (thermal energy) density $q = C_p\cdot\rho\cdot T$ ← temperature, ← mass density

heat flux $\vec{F} = -k\nabla T$

Heat conservation : $\frac{\partial q}{\partial t} + \nabla\cdot(-k\nabla T) = 0$

If ρ, C_p, k constant : $\frac{\partial T}{\partial t} = K\nabla^2 T$, $K = \frac{k}{C_p\rho}$

heat equation

• charge density ρ, current density $\vec{J}$:

Charge conservation : $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0$

Electromagnetism :

1. <u>Gauss' Law</u> revisited

Coulomb's Law (charge q at origin) $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{R}}{|\vec{R}|^3} = kq\frac{\hat{e}_r}{r^2}$, $k = \frac{1}{4\pi\varepsilon_0}$

(SI units : MKS)

ε_0 : permittivity of free space
8.854×10^{-12}
$C^2 N^{-1} m^{-2}$

We showed previously:

$$\iint_{S_a} \vec{E}\cdot\hat{n}\, dS = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{a^2}\cdot 4\pi a^2 = \frac{q}{\varepsilon_0}$$: S_a - sphere of radius a, enclosing origin

We also know $\nabla\cdot\vec{E} = 0$ except at $\vec{R} = \vec{0}$ ← in general: at location of charges

(spherical coordinates: $\vec{E} = \frac{kq}{r^2}\hat{e}_r \Rightarrow \nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{kq}{r^2}\cdot r^2\right) = 0$)

We can now use the divergence theorem to show Gauss' Law (carefully)

- $\iint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$: Q : <u>total</u> charge enclosed by the closed surface S

Since electric fields are additive, (and by translation invariance)
it is sufficient to show (for a single point charge q at the origin)

that
$$\underset{\partial V \nearrow S}{\iint} \vec{E}\cdot\hat{n}\,dS = \begin{cases} q/\varepsilon_0 & \text{if the surface } S \text{ encloses the origin} \\ 0 & \text{if } S \text{ does not enclose the origin} \end{cases}$$

<u>Proof</u>:
Let V be the volume bounded by $S = \partial V$.

- If the origin $(0,0,0) \notin V$ (ie V contains no charges), then

$$\iint_S \vec{E}\cdot\hat{n}\,dS \underset{\substack{\uparrow \\ \text{divergence} \\ \text{theorem}}}{=} \iiint_V \underbrace{\nabla\cdot\vec{E}}_{=0 \text{ away from charges.}}\,dV = 0$$

- If the origin is in V, $(0,0,0) \in V$, then we cannot apply the divergence theorem in V (since $\nabla\cdot\vec{E}$ is not continuous at the origin)
But consider a small sphere S_a [radius a] (centred at $(0,0,0)$) contained in V; then we can apply the divergence theorem to the region V_a between S_a and S ($\nabla\cdot\vec{E} = 0$ in V_a)

(note: on S_a the normal vector is $\hat{e}_r$, while the "outward normal" direction to V_a is $-\hat{e}_r$) :

$$0 = \iiint_{V_a} \nabla\cdot\vec{E}\,dV \underset{\substack{\uparrow \\ \text{divergence} \\ \text{theorem}}}{=} \iint_S \vec{E}\cdot\hat{n}\,dS - \iint_{S_a} \vec{E}\cdot\hat{n}\,dS$$

$$\Rightarrow \iint_S \vec{E}\cdot\hat{n}\,dS = \iint_{S_a} \vec{E}\cdot\hat{n}\,dS = q/\varepsilon_0$$ — as required: Gauss' Law for more general surfaces

Gauss' Law: integral form $\iint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$ ← total enclosed charge in volume V bounded by S

(continuous charge distribution) $= \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$ ρ: charge density

divergence theorem $\Rightarrow \iiint_V \nabla\cdot\vec{E}\, dV = \iiint_V \frac{\rho}{\varepsilon_0}\, dV$ but volume V is arbitrary

$\Rightarrow \boxed{\nabla\cdot\vec{E} = \rho/\varepsilon_0}$ differential form of Gauss' Law (free space).

2. The magnetic flux through any closed surface is zero (no magnetic monopoles)

integral form $\Rightarrow \iint_S \vec{B}\cdot\hat{n}\,dS = 0$ $\xRightarrow{\text{div. thm}} \iiint_V \nabla\cdot\vec{B}\,dV = 0$ for any volume V

differential form $\Rightarrow \boxed{\nabla\cdot\vec{B} = 0}$

3. Faraday's Law (of induction) $\vec{E}, \vec{B}$: time-dependent electric and magnetic fields

S : a surface with boundary curve C

$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{\ell}$: voltage around C = electromotive force (emf) in C

$\Phi = \iint_S \vec{B}\cdot d\vec{S}$: magnetic flux through S

Faraday's Law: A changing magnetic flux induces an electromotive force (voltage / potential) in C: $\mathcal{E} = -\frac{d\Phi}{dt}$

ie $\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$ ← integral form

Stokes' theorem $\Rightarrow \iint_S (\nabla\times\vec{E})\cdot d\vec{S} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$ $\Rightarrow \iint_S \left(\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S} = 0$ for every surface S

$\Rightarrow \boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$ differential form of Faraday's Law

4. Ampère's Law

(time-independent fields...)

C: a closed curve, bounding surface S

Ampère's Law: The circulation of the magnetic field about C is proportional to the enclosed current

μ_0 : permeability of free space $1.257 \times 10^{-6}\ NA^{-2}$ $= \frac{1}{\varepsilon_0 c^2}$

$$\oint_C \vec{B} \cdot d\vec{R} = \mu_0 I$$ ← total current spanned by curve C

$$= \mu_0 \iint_S \vec{J} \cdot \hat{n}\, dS$$

$\vec{J}$: current density

using Stokes' theorem

$$\Rightarrow \iint_S \left[\nabla \times \vec{B} - \mu_0 \vec{J}\right] \cdot d\vec{S} = 0$$ (curve C, surface S are arbitrary

$$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J} .$$

Note: Maxwell realized that this equation cannot be true for general space- and time-dependent fields, as it implies $\nabla \cdot \vec{J} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \vec{B}) = \frac{1}{\mu_0} \cdot 0 = 0$, while conservation of charge (the continuity equation) requires $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$.
Since $\nabla \cdot \vec{E} = \rho/\varepsilon_0$, we have $\frac{\partial \rho}{\partial t} = \varepsilon_0 \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t}\right)$ so the continuity equation implies $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$
Thus if we set $\vec{J}_T = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, then $\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}_T}$ is consistent with the equation of continuity

Maxwell's Equations (in free space)

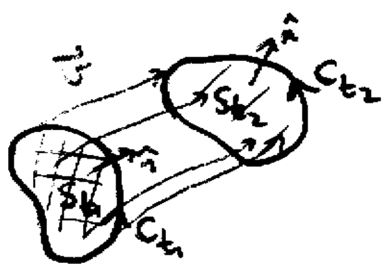
$\nabla \cdot \vec{E} = \rho/\varepsilon_0$	Gauss' Law
$\nabla \cdot \vec{B} = 0$	no magnetic monopoles
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's Law
$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$	Ampère's Law

Transport Theorems

[Consider integrals $\iint_S \vec{F}\cdot\hat{n}\,dS$, $\iiint_V f\,dV$ in which there could be time dependence of the fields $\vec{F}, f$, and also of the regions S, V: it is often necessary to compute the time derivative of a surface or volume integral when the region is in motion.]

Flux Transport Theorem

Vector field $\vec{F}(\vec{R}, t)$ (continuously differentiable)



S: an oriented surface, bounded by curve C, which moves through space — location of S, C changes with time

– denote the surface and curve at time t by S_t, C_t (or $S(t)$, $C(t)$)

$\vec{v}(\vec{R}, t)$: a velocity field on S that describes the pointwise motion on the surface

(ie points on S move according to the flow lines of $\vec{v}$: point $\vec{R}$ on S_{t_1} moves to $\vec{R}+d\vec{R} = \vec{R} + \vec{v}(\vec{R}, t_1)dt$ on S_{t_2} $(t_2 = t_1 + dt)$)

$$\Phi(t) = \iint_{S_t} \vec{F}(\vec{R}, t)\cdot d\vec{S}$$

Flux of $\vec{F}$ through surface S at time t

The flux Φ changes due to:
- changes in the field $\vec{F}$
- motion of surface S_t

Flux transport theorem

$$\boxed{\frac{d\Phi}{dt} = \iint_{S_t}\left(\frac{\partial\vec{F}}{\partial t} + (\nabla\cdot\vec{F})\vec{v}\right)\cdot d\vec{S} + \oint_{C_t}(\vec{F}\times\vec{v})\cdot d\vec{R}}$$

(Stokes' Theorem) ($\vec{v}\in C^1$ in a region ...)

$$= \iint_{S_t}\left(\frac{\partial\vec{F}}{\partial t} + (\nabla\cdot\vec{F})\vec{v} + \nabla\times(\vec{F}\times\vec{v})\right)\cdot d\vec{S}$$

Reynolds Transport Theorem

Scalar field $f(\vec{R},t)$
(continuously differentiable)

V : a moving region of integration;
at time t we have volume V_t bounded by surface S_t
(or $V(t)$, $S(t)$)

Points in V move according to the velocity field $\vec{v}(\vec{R},t)$
(describes changes in the volume with time)

Convective derivative: $\dfrac{Df}{Dt} = \dfrac{\partial f}{\partial t} + \vec{v}\cdot\nabla f$ [Notation: sometimes $\dfrac{df}{dt}$

(rate of change of f following the moving volume elements)

Reynolds transport theorem

$$\boxed{\frac{d}{dt}\iiint_{V_t} f\, dV = \iiint_{V_t}\left(\frac{Df}{Dt} + f\,\nabla\cdot\vec{v}\right)dV}$$

$$= \iiint_{V_t}\left(\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + f\,\nabla\cdot\vec{v}\right)dV$$

vector identity

$$= \iiint_{V_t}\left(\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v})\right)dV$$

alternative form $\Rightarrow$ (divergence theorem)

$$\boxed{\frac{d}{dt}\iiint_{V_t} f\, dV = \iiint_{V_t}\frac{\partial f}{\partial t}\,dV + \iint_{S_t} f\,\vec{v}\cdot d\vec{S}}$$

rate of change of volume integral of f in a moving volume = rate of change due to changes in f (fixed volume) + amount of f that is swept up through the moving sides (flux through surface)

Note on <u>convective derivative</u>:

- time-dependent field $f(\vec{R},t)$
- fluid motion with velocity field $\vec{v}$

$$\frac{\partial f}{\partial t} = \lim_{\Delta t\to 0} \frac{f(\vec{R}, t+\Delta t) - f(\vec{R},t)}{\Delta t}$$

– rate of change of f as measured by an observer at a <u>fixed</u> location $\vec{R}$: <u>Eulerian derivative</u> (partial derivative)

$$\frac{Df}{Dt} = \lim_{\Delta t\to 0} \frac{f(\vec{R}+\vec{v}\Delta t, t+\Delta t) - f(\vec{R},t)}{\Delta t}$$

– rate of change of f as measured by an observer <u>moving with the fluid</u>: <u>Lagrangian</u> derivative (fluid elements move with velocity $\frac{d\vec{R}}{dt} = \vec{v}$)

$$\frac{Df}{Dt} = \frac{d}{dt} f(\vec{R}(t), t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$= \nabla f(\vec{R}(t),t)\cdot\frac{d\vec{R}}{dt} + \frac{\partial f}{\partial t} \qquad \text{(Chain Rule!)}$$

$$\Rightarrow \boxed{\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f}$$

Heuristic "proof" (derivation) of <u>flux transport theorem</u>:

Notation: integration over spatial variables, for fixed time t_1

$$\frac{d\Phi}{dt} = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\iint_{S_{t_2}} \vec{F}(\cdot,t_2)\cdot d\vec{S} - \iint_{S_{t_1}} \vec{F}(\cdot,t_1)\cdot d\vec{S}\right]$$

$(t_2 = t_1 + \Delta t)$

fixed area, change in $\vec{F}$ with t

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\iint_{S_{t_2}} \vec{F}(\cdot,t_2) d\vec{S} - \iint_{S_{t_2}} \vec{F}(\cdot,t_1)\cdot d\vec{S}\right]$$

+

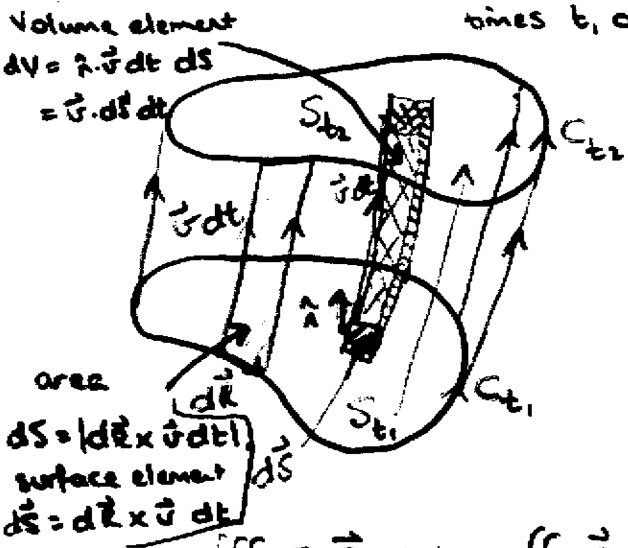
fixed vector field, change in S with t

$$+ \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\iint_{S_{t_2}} \vec{F}(\cdot,t_1)\cdot d\vec{S} - \iint_{S_{t_1}} \vec{F}(\cdot,t_1)\cdot d\vec{S}\right]$$

First term: $\lim\limits_{\Delta t \to 0} \iint\limits_{S_{t_2}} \left[\dfrac{\vec{F}(\cdot,t_2) - \vec{F}(\cdot,t_1)}{\Delta t}\right]\cdot d\vec{S} = \iint\limits_{S_{t_1}} \dfrac{\partial \vec{F}}{\partial t}(\cdot,t_1)\cdot d\vec{S}$

$t_2 = t_1 + \Delta t$ since $t_2 \to t_1$ as $\Delta t \to 0$

Second term: Consider the region D swept out by the surfaces between times t_1 and $t_2 = t_1 + dt$, the ends are S_{t_1} and S_{t_2}, the sides are the flow lines through the bounding curves, mapping C_{t_1} to C_{t_2}.



Let $\vec{G}(\vec{R})$ be a fixed (time-independent) vector field.

Apply the divergence theorem to $\vec{G}$ over the volume D:

$$\iiint\limits_D \nabla\cdot\vec{G}\,dV = \iint\limits_{\partial D} \vec{G}\cdot d\vec{S} = \iint\limits_{S_{t_2}} \vec{G}\cdot d\vec{S} - \iint\limits_{S_{t_1}} \vec{G}\cdot d\vec{S} + \iint\limits_{(\text{sides})} \vec{G}\cdot d\vec{S}$$

since if $\hat{n}$ is the surface normal on S_{t_1}, then $-\hat{n}$ is the outward normal to D.

$$\Rightarrow \iint\limits_{S_{t_2}} \vec{G}\cdot d\vec{S} - \iint\limits_{S_{t_1}} \vec{G}\cdot d\vec{S} = \iiint\limits_D \nabla\cdot\vec{G}\,dV - \iint\limits_{(\text{sides})} \vec{G}\cdot d\vec{S}$$

On the sides, surface element $d\vec{S} = d\vec{R}\times\vec{v}\,dt$ ($d\vec{R}$ taken along C_{t_1})

$$\Rightarrow \iint\limits_{(\text{sides})} \vec{G}\cdot d\vec{S} = dt \oint\limits_{C_{t_1}} \vec{G}\cdot d\vec{R}\times\vec{v} = -dt \oint\limits_{C_{t_1}} (\vec{G}\times\vec{v})\cdot d\vec{R}$$

change order in scalar triple product

The element of volume in D is $dV = dS\ \vec{v}\cdot\hat{n}\,dt$ "base area × height"
$= \vec{v}\cdot d\vec{S}\,dt$ ($d\vec{S}$ taken on S_{t_1})

$$\Rightarrow \iiint\limits_D \nabla\cdot\vec{G}\,dV = \iint\limits_{S_{t_1}} (\nabla\cdot\vec{G})\ \vec{v}\cdot\hat{n}\,dS\,dt = dt \iint\limits_{S_{t_1}} (\nabla\cdot\vec{G})\,\vec{v}\cdot d\vec{S}$$

Thus we compute (change in flux integral for fixed vector field, moving S)

$$\iint\limits_{S_{t_1+dt}} \vec{G}\cdot d\vec{S} - \iint\limits_{S_{t_1}} \vec{G}\cdot d\vec{S} = dt\left[\iint\limits_{S_{t_1}} (\nabla\cdot\vec{G})\vec{v}\cdot d\vec{S} + \oint\limits_{C_{t_1}} (\vec{G}\times\vec{v})\cdot d\vec{R}\right]$$

$$\Rightarrow \lim_{\Delta t\to 0} \frac{1}{\Delta t}\Big[\iint_{S_{t_2}} \vec{c}\cdot d\vec{S} - \iint_{S_{t_1}} \vec{c}\cdot d\vec{S}\Big] = \iint_{S_{t_1}} (\nabla\cdot\vec{c})\,\vec{v}\cdot d\vec{S} + \oint_{C_{t_1}} (\vec{c}\times\vec{v})\cdot d\vec{R}$$

$t_1+\Delta t$ rate of change of flux due to moving surface

Now let $\vec{c}(\vec{R})$ be the fixed vector field $\vec{F}(\vec{R},t_1)$, evaluated at time t_1

$$\vec{c}(\vec{R}) = \vec{F}(\vec{R},t_1)$$

so the second term is

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\iint_{S_{t_2}} \vec{F}(\cdot,t_1)\cdot d\vec{S} - \iint_{S_{t_1}} \vec{F}(\cdot,t_1)\cdot d\vec{S}\Big]$$

$$= \iint_{S_{t_1}} [(\nabla\cdot\vec{F})\,\vec{v}](\cdot,t_1)\cdot d\vec{S} + \oint_{C_{t_1}} (\vec{F}\times\vec{v})(\cdot,t_1)\cdot d\vec{R}$$

and combining the results
(let $t=t_1$) we find as desired:

Flux transport theorem

$$\frac{d\Phi}{dt} = \iint_{S_t}\Big[\frac{\partial\vec{F}}{\partial t} + (\nabla\cdot\vec{F})\,\vec{v}\Big]\cdot d\vec{S} + \oint_{C_t}(\vec{F}\times\vec{v})\cdot d\vec{R}$$

Heuristic "proof" (derivation) of Reynolds transport theorem

$$\frac{d}{dt}\iiint_{V_{t_1}} f\,dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\iiint_{V_{t_2}} f(\cdot,t_2)\,dV - \iiint_{V_{t_1}} f(\cdot,t_1)\,dV\Big]$$

$(t_2 = t_1+\Delta t)$

fixed volume, change in field f with t

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\iiint_{V_{t_2}} f(\cdot,t_2)\,dV - \iiint_{V_{t_2}} f(\cdot,t_1)\,dV\Big]$$

\+

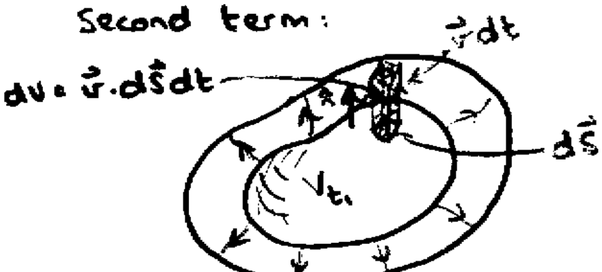
fixed scalar field, change in V with t

$$+\lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\iiint_{V_{t_2}} f(\cdot,t_1)\,dV - \iiint_{V_{t_1}} f(\cdot,t_1)\,dV\Big]$$

First term: $\displaystyle \lim_{\Delta t\to 0}\iiint_{V_{t_2}} \frac{f(\cdot,t_1+\Delta t)-f(\cdot,t_1)}{\Delta t}\,dV = \iiint_{V_{t_1}}\frac{\partial f}{\partial t}\,dV$ $\leftarrow$ $t_2\to t_1$ as $\Delta t\to 0$

$t_1+\Delta t$

Second term:



We need to quantify the effect of the change in volume from V_{t_1} to V_{t_2}.

Consider a small surface element $d\vec{S}$ on S_{t_1}.

The volume swept out by the velocity field $\vec{v}(\vec{R},t_1)$ in time dt is $dV = (\vec{v}\,dt)\cdot\hat{n}\,dS = \vec{v}\cdot d\vec{S}\,dt$

(could be negative) $\longrightarrow$

Thus for any fixed (time-independent) scalar field $g(\vec{R})$, we have

$$\underset{V_{t_2}}{\iiint} g\,dV - \underset{V_{t_1}}{\iiint} g\,dV = \underset{\substack{\text{(difference}\\ \text{in volumes)}}}{\iiint} g\,dV = dt \underset{S_{t_1}}{\iint} g\vec{v}\cdot d\vec{S}$$

t_1+dt

$$\text{(divergence theorem)} = dt \underset{V_{t_1}}{\iiint} \nabla\cdot(g\vec{v})\,dV$$

$$\Rightarrow \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\underset{V_{t_2}}{\iiint} g\,dV - \underset{V_{t_1}}{\iiint} g\,dV\right] = \underset{V_{t_1}}{\iiint} \nabla\cdot(g\vec{v})\,dV = \underset{S_{t_1}}{\iint} g\vec{v}\cdot d\vec{S}$$

$t_1+\Delta t$ — rate of change due to moving region

As before, now choose g to be the scalar field f evaluated at t_1:
$g(\vec{R}) = f(\vec{R}, t_1)$

Combining the two terms, and letting $t=t_1$, we find Reynolds' transport theorem

$$\frac{d}{dt}\underset{V_t}{\iiint} f\,dV = \underset{V_t}{\iiint}\left[\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v})\right]dV = \underset{V_t}{\iiint}\left[\frac{Df}{Dt} + f\,\nabla\cdot\vec{v}\right]dV$$

An important application of Reynolds' transport theorem is to the mathematical formulation of the equations of fluid motion.

eg $f = \rho(\vec{R}, t)$: mass density

Conservation of mass: $\displaystyle \frac{d}{dt}\underset{V_t}{\iiint}\rho\,dV = 0$ (total mass inside a region moving with the fluid is constant

$$\Leftrightarrow \underset{V_t}{\iiint}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right]dV = 0 \quad \text{- but } V_t \text{ is arbitrary}$$

$$\Leftrightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \quad \leftarrow \text{Continuity equation (as before) Eulerian form}$$

$$\Leftrightarrow \frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0 \quad \leftarrow \text{Lagrangian form}$$

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eg if $f = \rho v_i$: (ith component of momentum density)

then the rate of change of momentum in the moving fluid region V_t (ith component)

equals ith component of force ⟵ is $\displaystyle \frac{d}{dt}\underset{V_t}{\iiint}\rho v_i\,dV = \underset{V_t}{\iiint}\left[\frac{D(\rho v_i)}{Dt} + \rho v_i\,\nabla\cdot\vec{v}\right]dV$ - needed for formulation of Newton's Law