The Divergence Theorem (revisited)

Theorem: Let V be any region with the property that each straight line through any interior point parallel to a coordinate axis intersects the boundary in exactly two points; let the boundary S of V be a piecewise smooth, closed, oriented surface with unit normal directed outward from the domain.

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a continuously differentiable vector field on $\bar{V} = V \cup \partial V$ (ie $\vec{F}$ and its partial derivatives are continuous)

Let $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

Then

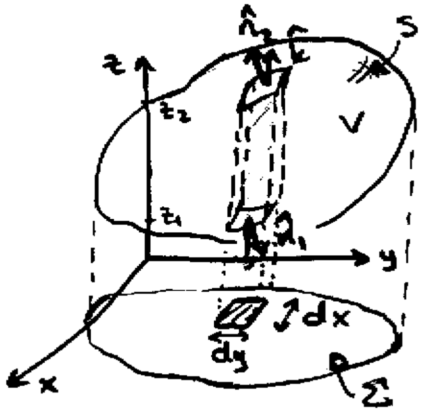
$$\boxed{\iiint_V \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S}} = \iint_S \vec{F}\cdot\hat{n}\,dS$$

Proof: Unit normal $\hat{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k} = \underbrace{\cos\alpha}_{\hat{n}\cdot\hat{\imath}}\,\hat{\imath} + \underbrace{\cos\beta}_{\hat{n}\cdot\hat{\jmath}}\,\hat{\jmath} + \underbrace{\cos\gamma}_{\hat{n}\cdot\hat{k}}\,\hat{k}$

In component form, the divergence theorem is

$$\iiint_V \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dV = \iint_S \left[F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma\right] dS$$

We will show $\iiint_V \frac{\partial F_3}{\partial z}\,dV = \iint_S F_3\,\hat{k}\cdot\hat{n}\,dS = \iint_S F_3\cos\gamma\,dS \left[= \iint_S F_3\,dx\,dy\right]$



— By our assumptions on the region V, a vertical line intersects S at two points; for each x,y, the region is $z_1(x,y) \le z \le z_2(x,y)$

Let Σ be the projection of the surface S onto the x-y plane.

We can evaluate the volume integral by first integrating along a vertical column.

Then $\displaystyle\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iint_\Sigma \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\, dx\, dy$

$$\overset{\text{Fundamental Theorem of Calculus}}{=} \iint_\Sigma \left[F_3(x,y,z_2(x,y)) - F_3(x,y,z_1(x,y))\right] dx\, dy$$

- Now on the upper surface $z = z_2(x,y)$, the outward normal $\hat{n}_2$ points upward, so $\hat{n}_2 \cdot \hat{k} > 0$, and the element of surface area is

$$dS = \frac{dx\,dy}{|\hat{n}_2 \cdot \hat{k}|} = \frac{dx\,dy}{\hat{n}_2 \cdot \hat{k}} \Rightarrow \underbrace{dx\,dy}_{\text{projection onto x-y plane}} = \hat{n}_2 \cdot \hat{k}\, dS = \cos\gamma\, dS$$

$$\Rightarrow \iint_\Sigma F_3(x,y,z_2(x,y))\,dx\,dy = \iint_{\substack{\text{upper}\\\text{surface}}} F_3\, \hat{k}\cdot\hat{n}\, dS = \iint_{\substack{\text{upper}\\\text{surface}}} F_3 \cos\gamma\, dS$$

On the lower part of the surface $z = z_1(x,y)$, the outward normal $\hat{n}_1$ points downward, so $\hat{n}_1 \cdot \hat{k} < 0$, and $dx\,dy = |\hat{n}_1 \cdot \hat{k}|\,dS = -\hat{n}_1\cdot\hat{k}\, dS$

$$\Rightarrow -\iint_\Sigma F_3(x,y,z_1(x,y))\,dx\,dy = \iint_{\substack{\text{lower}\\\text{surface}}} F_3\, \hat{k}\cdot\hat{n}\, dS = \iint_{\substack{\text{lower}\\\text{surface}}} F_3 \cos\gamma\, dS$$

- Combining these results, we find

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iint_S F_3\, \hat{k}\cdot\hat{n}\, dS \left[= \iint_S F_3 \cos\gamma\, dS = \iint_S F_3\, dx\, dy\right]$$

Similarly

$$\iiint_V \frac{\partial F_1}{\partial x}\, dV = \iint_S F_1\, \hat{i}\cdot\hat{n}\, dS \left[= \iint_S F_1 \cos\alpha\, dS = \iint_S F_1\, dy\, dz\right]$$

$$\iiint_V \frac{\partial F_2}{\partial y}\, dV = \iint_S F_2\, \hat{j}\cdot\hat{n}\, dS \left[= \iint_S F_2 \cos\beta\, dS = \iint_S F_2\, dz\, dx\right]$$

Adding these three results together, we obtain the <u>divergence theorem</u>. ⊠

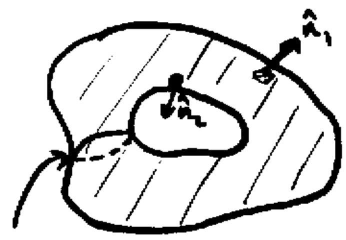
In our proof above, we used the assumption that V is a region in which each line parallel to a coordinate axis crosses the boundary in two points. In a similar way to the proof of Green's theorem we can easily extend the divergence theorem to any region which can be broken up into a finite number of such "elementary" regions (the result for more general regions can then be obtained by a limiting process) – since the surface integrals over any added interior surfaces will cancel

eg dumbell space

introduce a surface S' here

- then $V = V_1 \cup V_2$
- the contribution to the total surface integral due to S' cancels, since $\hat{n}_1 dS_1 = -\hat{n}_2 dS_2$ $[d\vec{S}_1 = -d\vec{S}_2]$ on S'.

eg region between two closed surfaces



introduce surface...

Thus

$$\iiint_V \operatorname{div} \vec{F}\, dV = \iint_S \vec{F}\cdot\hat{n}\, dS$$

holds for general regions V with piecewise smooth oriented surfaces S.

Divergence as flux per unit volume:

Assume $\vec{F}$ is continuously differentiable ($\Rightarrow \nabla\cdot\vec{F}$ continuous)

Let the region V with boundary $S = \partial V$ be a small region around the point P.

Then $\quad \iiint_V \operatorname{div}\vec{F}\, dV \approx [\operatorname{div}\vec{F}(P)].\operatorname{vol}(V)$ (mean value theorem)

- more precisely $\quad \underset{\text{evaluated at } P}{\operatorname{div}\vec{F}} = \lim_{V\to 0} \dfrac{\iiint_V \operatorname{div}\vec{F}\, dV}{V}$

By the divergence theorem we thus have

$$\boxed{\operatorname{div}\vec{F} = \lim_{V\to 0} \frac{\iint_S \vec{F}\cdot\hat{n}\, dS}{V}}$$

– divergence is flux per unit volume

Note that in the proof of the divergence theorem, we proved results about individual components, so we can combine these formulas to obtain other results

eg for a scalar field f,

$$\iiint_V \nabla f \, dV \overset{\text{def}}{=} \hat{\imath} \iiint_V \frac{\partial f}{\partial x} dV + \hat{\jmath} \iiint_V \frac{\partial f}{\partial y} dV + \hat{k} \iiint_V \frac{\partial f}{\partial z} dV$$

from (proof of) divergence theorem →

$$= \hat{\imath} \iint_S f \, \hat{\imath}\cdot\hat{n} \, dS + \hat{\jmath} \iint_S f \, \hat{\jmath}\cdot\hat{n} \, dS + \hat{k} \iint_S f \, \hat{k}\cdot\hat{n} \, dS$$

$$= \iint_S f \left[\underbrace{(\hat{n}\cdot\hat{\imath})}_{\cos\alpha}\hat{\imath} + \underbrace{(\hat{n}\cdot\hat{\jmath})}_{\cos\beta}\hat{\jmath} + \underbrace{(\hat{n}\cdot\hat{k})}_{\cos\gamma}\hat{k} \right] dS = \iint_S f \, \hat{n} \, dS$$

$$\Rightarrow \boxed{\iiint_V \nabla f \, dV = \iint_S f \, \hat{n} \, dS} = \iint_S f \, d\vec{S}$$

<u>Integration by parts</u>:

Recall, in one dimension: $f, g \in C^1[a,b]$ ← continuously differentiable on an interval $[a,b]$

$$\frac{d}{dx}(fg) = (fg)' = f'g + fg'$$

$$\Rightarrow \int_a^b (fg)' dx = \int_a^b f'g \, dx + \int_a^b fg' \, dx$$

but by the fundamental theorem of calculus, $\int_a^b (fg)' dx = fg\big|_a^b$

$$= (fg)(b) - (fg)(a)$$

$$\Rightarrow \int_a^b f g' \, dx = fg\big|_a^b - \int_a^b f'g \, dx$$: integration by parts

[or: $\left.\begin{array}{ll} u = f(x) & du = f'(x)dx \\ v = g(x) & dv = g'(x)dx \end{array}\right\}$; $\int_a^b u \, dv = uv\big|_a^b - \int_a^b v \, du$

indefinite integral → $\int u \, dv = uv - \int v \, du$]

⇒ integration by parts follows from:
- product rule &
- fundamental theorem of calculus

Integration by parts in higher dimensions?

Green's Formulas

Smooth scalar fields ϕ, ψ, vector field $\vec{F}$

Product rule (vector identity) $\nabla\cdot(\phi\vec{F}) = \phi\,\nabla\cdot\vec{F} + \vec{F}\cdot\nabla\phi$

Integrate over volume V with piecewise smooth oriented surface S

$$\iiint_V \nabla\cdot(\phi\vec{F})\,dV = \iiint_V \phi\,\nabla\cdot\vec{F}\,dV + \iiint_V \vec{F}\cdot\nabla\phi\,dV$$

but by the divergence theorem $\iiint_V \nabla\cdot(\phi\vec{F})\,dV = \iint_{S \leftarrow \partial V} \phi\vec{F}\cdot d\vec{S}$

$$\Rightarrow \iiint_V \phi\,\nabla\cdot\vec{F}\,dV = \iint_S \phi\vec{F}\cdot d\vec{S} - \iiint_V \vec{F}\cdot\nabla\phi\,dV$$

Let $\vec{F} = \nabla\psi \Rightarrow \nabla\cdot\vec{F} = \nabla^2\psi$

Then

$$\boxed{\iiint_V \phi\,\nabla^2\psi\,dV = \iint_S \phi\nabla\psi\cdot d\vec{S} - \iiint_V \nabla\phi\cdot\nabla\psi\,dV}$$

Green's First Formula

"integration by parts"

Exchange ϕ and ψ: similarly

$$\iiint_V \psi\,\nabla^2\phi\,dV = \iint_S \psi\,\nabla\phi\cdot d\vec{S} - \iiint_V \nabla\psi\cdot\nabla\phi\,dV$$

Subtract (the last terms cancel):

$$\boxed{\iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dV = \iint_S [\phi\nabla\psi - \psi\nabla\phi]\cdot d\vec{S}}$$

Green's Second Formula

(equivalently: $\iiint_V \phi\nabla^2\psi\,dV = \iint_S \phi\nabla\psi\cdot d\vec{S} - \iint_S \psi\,\nabla\phi\cdot d\vec{S} + \iiint_V (\nabla^2\phi)\,\psi\,dV$

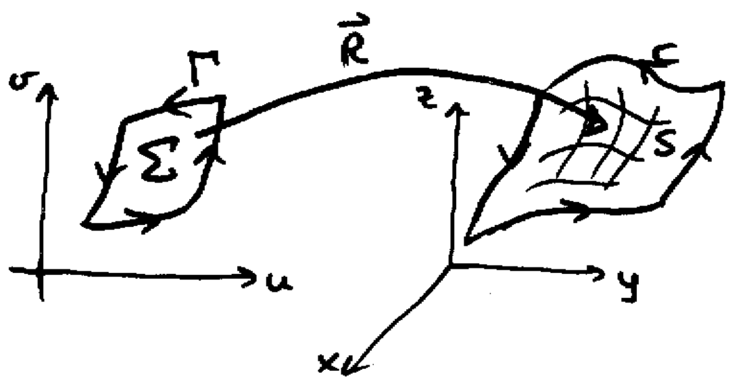
"integration by parts twice"]

Note: $\iint_S \phi\,\nabla\psi\cdot d\vec{S} = \iint_S \phi\nabla\psi\cdot\hat{n}\,dS = \iint_S \phi\frac{\partial\psi}{\partial n}\,dS$ where $\frac{\partial\psi}{\partial n} = \hat{n}\cdot\nabla\psi$

directional ("normal") derivative

Stokes' Theorem (revisited)

Theorem: Let S be any smooth oriented surface bounded by a (piecewise) smooth closed curve C whose orientation is consistent with that of S. Assume that the surface S can be parametrized by a one-to-one parametrization $\vec{R} = \vec{R}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$.
Here $\vec{R} : \Sigma \to S$, ie $\vec{R}$ maps a region $\Sigma \subset \mathbb{R}^2$ of the u-v plane to S, where Σ, with boundary Γ, is a region to which Green's theorem in the plane applies. Also assume that x, y and z are twice continuously differentiable functions of u, v on Σ (so that mixed partial derivatives are equal), with $\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}$ pointing in the direction of the normal $\hat{n}$ of S.



$\left.\begin{array}{l} \vec{R} : \Sigma \to S \\ \vec{R} : \Gamma \to C \end{array}\right\}$ preserving orientation

[Note: the theorem can be extended to more general piecewise smooth surfaces.]

Let $\vec{F} = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$ be a continuously differentiable vector field on $\bar{S} = S \cup C$ (that is, $\vec{F}(\vec{R}(u,v))$ is continuously differentiable for $(u,v) \in \Sigma \cup \Gamma$).

Let
$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

Then
$$\boxed{\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{R}} \qquad \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot \vec{T}\, ds$$

the circulation of $\vec{F}$ around C equals the flux of its curl through S.

Proof. We express both integrals in parametric form; we integrate the l.h.s. over the region Σ in the x-y plane, and the r.h.s over its boundary Γ, and apply Green's theorem.

Note: in terms of the parametization $\vec{R} = \vec{R}(u,v)$, we have

$$d\vec{S} = \hat{n}\,dS = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\,du\,dv$$

and

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u}\,du + \frac{\partial \vec{R}}{\partial v}\,dv \quad .$$

Furthermore, by the chain rule,

$$\frac{\partial \vec{F}}{\partial u} = \frac{\partial}{\partial u}\vec{F}(\vec{R}(u,v)) = \frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial u}$$

$$= \left(\frac{\partial \vec{R}}{\partial u}\cdot\nabla\right)\vec{F} \qquad \left[\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} = \frac{\partial \vec{R}}{\partial u}\cdot\nabla\right]$$

and similarly

$$\frac{\partial \vec{F}}{\partial v} = \left(\frac{\partial \vec{R}}{\partial v}\cdot\nabla\right)\vec{F} \quad .$$

We compute

$$\oint_C \vec{F}\cdot d\vec{R} = \oint_\Gamma \vec{F}(\vec{R}(u,v))\cdot\left[\frac{\partial \vec{R}}{\partial u}du + \frac{\partial \vec{R}}{\partial v}dv\right]$$

$$= \oint_\Gamma \left(\vec{F}\cdot\frac{\partial \vec{R}}{\partial u}\right)du + \left(\vec{F}\cdot\frac{\partial \vec{R}}{\partial v}\right)dv$$

Green's theorem in the u-v plane $\left[\oint_\Gamma G_1\,du + G_2\,dv = \iint_\Sigma\left(\frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v}\right)du\,dv\right]$

$$= \iint_\Sigma \left[\frac{\partial}{\partial u}\left(\vec{F}\cdot\frac{\partial \vec{R}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\vec{F}\cdot\frac{\partial \vec{R}}{\partial u}\right)\right]du\,dv$$

product rule

$$= \iint_\Sigma \left[\frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{R}}{\partial v} + \cancel{\vec{F}\cdot\frac{\partial^2 \vec{R}}{\partial u\partial v}} - \frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{R}}{\partial u} - \cancel{\vec{F}\cdot\frac{\partial^2 \vec{R}}{\partial v\partial u}}\right]du\,dv$$

equality of mixed partials

$$= \iint_\Sigma \left[\frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{R}}{\partial v} - \frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{R}}{\partial u}\right]du\,dv$$

chain rule

$$= \iint_\Sigma \left[\frac{\partial \vec{R}}{\partial v}\cdot\left(\frac{\partial \vec{R}}{\partial u}\cdot\nabla\right)\vec{F} - \frac{\partial \vec{R}}{\partial u}\cdot\left(\frac{\partial \vec{R}}{\partial v}\cdot\nabla\right)\vec{F}\right]du\,dv$$

Similarly, the other integral in parametric form is

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_\Sigma (\nabla \times \vec{F})(\vec{R}(u,v)) \cdot \left(\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right) du\, dv$$

We can thus prove Stokes' theorem by showing that the integrands in the above two integrals are the same, that is, by establishing the vector identity

$$\frac{\partial \vec{R}}{\partial v} \cdot \left(\frac{\partial \vec{R}}{\partial u} \cdot \nabla \right) \vec{F} - \frac{\partial \vec{R}}{\partial u} \cdot \left(\frac{\partial \vec{R}}{\partial v} \cdot \nabla \right) \vec{F} = \left(\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right) \cdot \nabla \times \vec{F}$$

We can show this using operator identities, or directly using tensor notation (where $\vec{R} = x_i(u,v)\,\hat{e}_i$, $\vec{F} = F_i(x_1,x_2,x_3)\,\hat{e}_i$) :

The expression on the l.h.s. is

$$\frac{\partial x_j}{\partial v} \left(\frac{\partial x_k}{\partial u} \frac{\partial}{\partial x_k} \right) F_j - \frac{\partial x_j}{\partial u} \left(\frac{\partial x_k}{\partial v} \frac{\partial}{\partial x_k} \right) F_j = \frac{\partial x_k}{\partial u} \frac{\partial x_j}{\partial v} \frac{\partial F_j}{\partial x_k} - \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} \frac{\partial F_j}{\partial x_k}$$

while the r.h.s. is

$$\begin{aligned}
\left(\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} \right)_i (\nabla \times \vec{F})_i &= \varepsilon_{ijk} \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} \varepsilon_{i\ell m} \frac{\partial F_m}{\partial x_\ell} \\
&= \varepsilon_{ijk}\, \varepsilon_{i\ell m} \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} \frac{\partial F_m}{\partial x_\ell} \\
&= (\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}) \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} \frac{\partial F_m}{\partial x_\ell} \\
&= \frac{\partial x_\ell}{\partial u} \frac{\partial x_m}{\partial v} \frac{\partial F_m}{\partial x_\ell} - \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} \frac{\partial F_j}{\partial x_k} = \text{l.h.s}
\end{aligned}$$

(with $\ell = k$, $m = j$)

With this vector identity, we have proved Stokes' theorem. ☒

Curl as circulation per unit area:

Assume $\vec{F}$ is continuously differentiable ($\Rightarrow \nabla \times \vec{F}$ has continuous components)

Let the surface S, with area A, normal $\hat{n}$, boundary $C = \partial S$, be a small surface of arbitrary shape containing the point P.

Then $\operatorname{curl} \vec{F} \cdot \hat{n} = \lim_{A \to 0} \dfrac{\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, dS}{A}$ (by continuity of $\nabla \times \vec{F}$)

component of curl $\vec{F}$ normal to surface S, at P

So by Stokes' theorem, $\boxed{\operatorname{curl} \vec{F} \cdot \hat{n} = \lim_{A \to 0} \dfrac{\oint_C \vec{F} \cdot d\vec{R}}{A}}$ - curl is circulation per unit area