Problem 1: 2.2.7

Find the fixed points, determine their stability, for
\[
\frac{dx(t)}{dt} = \cos x - e^x
\]

It is easier to visualize plotting the curves of \(\cos x\) and \(e^x\) separately:
\[
> \text{plot}(\{\cos(x), \exp(x)\}, x = -5*\text{Pi}..\text{Pi}, y = -2..2);
\]
The fixed points are at the intersections of these two curves. Let's find the first few numerically, specifying the intervals for the numerical solution, which we know from the behaviour of \( \cos x \):

\[
\begin{align*}
\text{x0} &= \text{fsolve}(\exp(x) - \cos(x), x, -1..1); \\
\text{x1} &= \text{fsolve}(\exp(x) - \cos(x), x, -\pi..-1); \\
\text{x2} &= \text{fsolve}(\exp(x) - \cos(x), x, -2\pi..-\pi); \\
\text{x3} &= \text{fsolve}(\exp(x) - \cos(x), x, -3\pi..-2\pi); \\
\text{x4} &= \text{fsolve}(\exp(x) - \cos(x), x, -4\pi..-3\pi);
\end{align*}
\]

\[
\begin{align*}
\text{x0} &= 0.0.0. \\
\text{x1} &= -1.292695719 \\
\text{x2} &= -4.721292759 \\
\text{x3} &= -7.853593280 \\
\text{x4} &= -10.99559106
\end{align*}
\]

We observe that the roots \( x_k \) for \( 2 \leq k \) are close to the zeros of \( \cos x \), since \( e^x \) is small:

\[
\begin{align*}
\text{evalf(seq}(-(k-1/2)*\pi, k=2..4)) \\
-4.712388981, -7.853981635, -10.99557429
\end{align*}
\]
From the sign of $e^x - \cos x$, we can see that the fixed points $x_k$ are stable if $k$ is odd, and unstable if $k$ is even.

We can deduce the qualitative behaviour of the solutions from the fixed points and their stability.

Direct numerical integration using Maple gives:

```maple
eqp1 := diff(x(t),t) = exp(x(t)) - cos(x(t));
initconds :=
[ [x(0)=1], [x(0)=0], [x(0)=-1], [x(0)=-2], [x(0)=-4], [x(0)=x2], [x(0)=-5],
  [x(0)=-7],
  [x(0)=-8] ];
DEplot(eqp1,x(t),t=0..8,x=-9..2,initconds,linecolor=black);
```

A closed-form analytical solution is not available, since it would require the integration of $1/(e^x - \cos x)$.

**Problem 2: 2.2.8**

We seek a dynamical system yielding the given flow. A possible answer is given by

$$\frac{dx(t)}{dt} = (x+1)^2 x(x-2)$$

```maple
eqp2 := diff(x(t),t) = (x(t)+1)^2 * x(t) * (x(t)-2);
initconds2 :=
[ [x(0)=-1.5], [x(0)=-1], [x(0)=-0.8], [x(0)=0], [x(0)=1.2],
  [x(0)=2], [x(0)=2.001] ];
DEplot(eqp2,x(t),t=0..5,x=-2..3,initconds2,linecolor=black);
```
Problem 3: 2.2.10 - Fixed Points

We seek examples of first-order dynamical systems \( \frac{dx}{dt} = f(x) \) satisfying certain conditions:

(a) Every real number is a fixed point:
\[
\frac{dx}{dt} = 0
\]

(b) Every integer is a fixed point, and there are no others:
\[
\frac{dx}{dt} = \sin(\pi x)
\]

(c) There are no examples in which there are exactly three fixed points, and all are stable. In fact, we cannot even have two stable fixed points adjacent to each other. This is easily seen by drawing a picture; but we can also argue as follows: Suppose \( x_1 \) and \( x_2 \) are two adjacent fixed points. Then, from the assumptions that \( f(x) \) is continuous, and that \( f(x) \) is nonzero between \( x_1 \) and \( x_2 \) (there are no fixed points between \( x_1 \) and \( x_2 \)), we must have that \( f(x) \) has one sign between \( x_1 \) and \( x_2 \). Suppose without loss of generality that \( f(x) > 0 \) in this interval. Then for any initial condition \( x_0 \) in this interval, the solution must be increasing (\( dx/dt > 0 \)), and must approach \( x_2 \) as \( t \to \infty \), consistent with the assumption that \( x_2 \) is stable. However, this contradicts the assumption that \( x_1 \) is stable, since we can choose the initial point \( x_0 \) as close to (but above) \( x_1 \) as we like.

(d) An equation with no fixed points is
\[
\frac{dx}{dt} = 1
\]

(e) A simple example of an equation with precisely 100 fixed points is
\[
\frac{dx}{dt} = (x - 1)(x - 2) \ldots (x - 99)(x - 100)
\]

**Problem 4: 2.2.13 - Skydiver**

> \( m := 'm': \ g := 'g': \ k := 'k': \)

\[
\text{eqp3} := \text{diff}(v(t), t) = g - \frac{k v(t)^2}{m};
\]

The command `odeadvisor` indicates the solution method for this first-order ODE.

> `odeadvisor(eq)`;

\[
\text{[\_quadrature]}\]

To find the solution, we separate variables, and integrate; the integration of \((a^2 - v^2)^{-1}\) (where \(a = \sqrt{\frac{gm}{k}}\)) is best performed using partial fractions. Maple can also solve this equation analytically:

\[
\text{dsolve(eq)};
\]

\[
v(t) = \frac{\tanh\left(\frac{\sqrt{gmk} \left( t + C1 \right)}{m}\right) \sqrt{gmk}}{k}
\]

In fact, this solution is only valid if \(m, g\) and \(k\) are all positive, and if \(-\sqrt{\frac{mg}{k}} < v(t)\). It seems that Maple automatically made these assumptions; in this case they are justified, but this example shows that in general Maple's analytical solutions are not always reliable: do the calculations by hand, and show your working! (you can use Maple to check, if you like). That is, the solution produced by Maple is only the general solution for \(a < v(0)\) (the problem is to take care with absolute value signs ...). We can find the particular solution satisfying the initial condition \(v(0) = 0\):

> `solp3 := rhs(dsolve({eqp3,v(0)=0}))`; 

\[
solp3 := \frac{\tanh\left(\frac{\sqrt{gmk} t}{m}\right) \sqrt{gmk}}{k}
\]

Now we can use Maple to find the asymptotic behaviour:

> `limit(solp3,t=infinity)`;

\[
\lim_{t \to \infty} \frac{\tanh\left(\frac{\sqrt{gmk} t}{m}\right) \sqrt{gmk}}{k}
\]

Evidently, now (finally!) Maple is concerned about the sign of the variables. Let's try specifying that all variables are positive:

> `limit(solp3,t=infinity) assuming (g>0,m>0,k>0);`
This gives the correct terminal velocity. We can write the formula for \( v(t) \) in terms of the terminal velocity \( V \):

\[
\text{vsol} := \text{simplify}(\text{subs}(m=k*V^2/g, \text{solp3})) \quad \text{assuming} \quad (k>0, V>0);
\]

\[
\text{vsol} := \tanh\left(\frac{t g}{V}\right) V
\]

This answer is much more easily obtained by the graphical method. We plot \( \frac{dv}{dt} \) against \( v \) (choosing some values of the variables):

\[
g := 10: m := 0.1: k := 1:
\]

\[
\text{plot}(g - k*v*v/m, v=-1.5..1.5);
\]

\[
m := 'm': g := 'g': k := 'k':
\]

Now let's use the numbers given:

The average velocity is (in ft/sec)

\[
V_{\text{avg}} := \frac{31400-2100}{116} \quad \text{evalf}(V_{\text{avg}});
\]

\[
V_{\text{avg}} := \frac{7325}{29}
\]

252.5862069

The distance travelled as a function of time is \( s(t) \) satisfying \( \frac{ds}{dt} = v \) and \( s(0) = 0 \).

\[
s_1 := \text{int}(\text{vsol}, t) \quad \text{assuming} \quad (V>0, g>0);
\]

\[
s_1 := -\frac{1}{2} \frac{V^2 \ln\left(\frac{t g}{V} - 1\right)}{g} - \frac{1}{2} \frac{V^2 \ln\left(\frac{t g}{V} + 1\right)}{g}
\]

This solution doesn't look too correct: it is giving negative arguments of the \( \ln \) function (and is
thus complex-valued). Let's try to help Maple a bit, by performing the appropriate substitution by hand...

\begin{align*}
  s_2 & := \int (vsol, t) \text{ assuming } (V>0, g>0); \\
  > s_3 & := \text{value} (\text{student}[\text{changevar}](\tau=t*\frac{g}{V}, s_2, \tau)); \\
  > s & := \text{subs}(\tau=t*\frac{g}{V}, s_3);
\end{align*}

\begin{align*}
  s_2 & := \int \tanh\left(\frac{t g}{V}\right) \frac{V}{g} dt \\
  s_3 & := \frac{\frac{V^2 \ln(\cosh(\tau))}{g}}{\frac{V^2 \ln\left(\cosh\left(\frac{t g}{V}\right)\right)}{g}} \\
  s & := \frac{g}{V^2} \frac{\ln\left(\frac{3735.2}{V}\right)}{\ln\left(\frac{3735.2}{V}\right)}
\end{align*}

\begin{align*}
  > g & := 32.2; t := 116; s; \text{dist} := 31400-2100; \\
  \text{solve}(s=\text{dist}, V);
\end{align*}

In the command solve, Maple attempts an analytical solution; in this case it gets it wrong (I'm not sure why; but the given value is the average velocity computed previously, which cannot also be the terminal velocity). For a problem with purely floating-point solutions, we should use fsolve (and look for the positive solution):

\begin{align*}
  V_{\text{term}} & := \text{fsolve}(s=\text{dist}, V, V=0..\text{infinity}); \\
  V_{\text{term}} & := 265.6854815
\end{align*}

From this value of the terminal velocity, we can compute the drag constant \(k\). Note that the weight (in pounds) is \(mg\).

\begin{align*}
  \text{weight} & := 261.2; \\
  \text{kdrag} & := \text{solve}(\sqrt{\text{weight}/k}=V_{\text{term}}, k); \\
  V_{\text{term}} & := 265.6854815
\end{align*}

\begin{align*}
  > m & := 'm'; g := 'g'; k := 'k'; s := 's'; t := 't';
\end{align*}

\section*{Problem 5: 2.3.2 - Autocatalysis}

The fixed points are readily found to be 0 and \(\frac{k_1 a}{k_(-1)}\)

\begin{align*}
  > \text{fp4} & := k_1 a x - k_(-1) x^2; \\
  \text{solve}(\text{fp4}, x);
\end{align*}

\begin{align*}
  \text{fp4} & := k_1 a x - k_(-1) x^2
\end{align*}
$x = 0$ is unstable, the other fixed point is stable.

We can do a quick graphical analysis, and plot some typical solutions, if we assume values for the constants:

\[
a := 1; \quad k_1 := 1; \quad km_1 := 1;
\]

```maple
plot(fp4,x=-0.5..1.5);
```

Problem 6: 2.4.7 - Pitchfork bifurcation

```maple
f6 := (x,a) -> a*x - x^3;
```
We plot the three vector fields next to each other, using the array function:

\[
p6a := \text{plot}(f6(x,-1),x=-1.5..1.5,y=-2..2,\text{tickmarks}=[0,0]):
\]
\[
p6b := \text{plot}(f6(x,0),x=-1.5..1.5,y=-2..2,\text{tickmarks}=[0,0]):
\]
\[
p6c := \text{plot}(f6(x,1),x=-1.5..1.5,y=-2..2,\text{tickmarks}=[0,0]):
\]
\[
> \text{plots6} := \text{array}(1..1,1..3):
\]
\[
\text{plots6}[1,1]:=p6a: \text{plots6}[1,2]:=p6b: \text{plots6}[1,3]:=p6c:
\]
\[
\text{display}(\text{plots6});
\]

For \(a \leq 0\), there is a unique fixed point at \(x = 0\), which is stable; for \(a < 0\) this is found by linear stability analysis (since \(f'(0) = a < 0\)), while for \(a = 0\), linear stability analysis does not prove stability (decay towards the origin is slower than exponential - see the next problem), but a look at the plot of \(-x^3\) shows that the origin is stable.

If \(0 < a\), there are three fixed points, at \(x = 0\), \(x = -\sqrt{a}\) and \(x = \sqrt{a}\). Now \(f'(0) = a\) is positive, so the origin is unstable, while the other two fixed points are stable, with \(f' = -2a\). This is also apparent from the graphs.

\section*{Problem 7: 2.4.9 - Critical slowing down}
Reset variables:
\[
x0 := 'x0':
\]
\[
> \text{eqp7} := \frac{\text{diff}(x(t),t)}{} = -x(t)^3;
\]
\[
\text{eqp7} := \frac{dx(t)}{dt} = -x^3(t)
\]
Find the analytical solution with arbitrary initial condition:
\[
xsa := \text{dsolve}([\text{eqp7},x(0)=x0],x(t)) \text{ assuming } x0>0;
\]
\[
xsb := \text{dsolve}([\text{eqp7},x(0)=x0],x(t)) \text{ assuming } x0<0;
\]
\[
xsz := \text{dsolve}([\text{eqp7},x(0)=0],x(t));
\]
\[ x_{sa} := x(t) = \frac{1}{\sqrt{2t + \frac{1}{x_0^2}}} \]

\[ x_{sb} := x(t) = -\frac{1}{\sqrt{2t + \frac{1}{x_0^2}}} \]

\[ x_{sz} := x(t) = 0 \]

So the solutions approach zero for arbitrary initial conditions; however, the decay is proportional to \( \frac{1}{\sqrt{t}} \), not exponential.

We plot the solutions of this equation and of \( \frac{dx}{dt} = -x \) on the same graph:

\[ \text{lineq} := \text{diff}(x(t), t) = -x(t) ; \]
\[ \text{linsoln} := \text{dsolve}([\text{lineq}, x(0)=10], x(t)) ; \]
\[ \text{critsoln} := \text{dsolve}([\text{eqp7}, x(0)=10], x(t)) ; \]

\[ \text{linsoln} := x(t) = 10 e^{(-t)} \]

\[ \text{critsoln} := x(t) = \frac{1}{\sqrt{2t + \frac{1}{100}}} \]

\[ \text{plot}([\text{rhs(linsoln)}, \text{rhs(critsoln)]}, t=0..10); \]
Note that the solution to $\frac{dx}{dt} = -x^3$ decays much more rapidly initially, but then slows down once $x < 1$.

**Problem 8: 2.5.1 - Reaching origin in finite time**

The origin $x = 0$ is a stable fixed point for any real $0 < c$. We plot a few representative vector fields:

```maple
>
plot(-x^(1/2),x=0..2,tickmarks=[0,0]);
plot(-x^1,x=0..4,-4..0.5,tickmarks=[0,0]);
plot(-x^2,x=0..2,y=-4..0.5,tickmarks=[0,0]);
```
We know that for \( c = 1 \), the decay towards the origin is exponential, and \( x \) approaches 0 asymptotically. When \( 1 < c \), the decay is slower than exponential, as we derived in Problem 7. So the only possibility for the solution to decay to zero in finite time is for \( c < 1 \).

The time taken from \( x = 1 \) to \( x = 0 \) is

\[
T = \int (-1/x^c, x=1..0);
\]

When \( 1 < c \), the limit diverges; when \( c = 1 \), \( T \) is also infinite (\( T = -\lim \ln x \)). When \( c < 1 \), the time
Problem 9: 2.5.2 - Blow-up

We know that solutions \( y(t) \) of \( \frac{dy}{dt} = 1 + y^2 \) blow up in finite time. Now for \( 1 < x \), the solutions \( x(t) \) of \( \frac{dx}{dt} = 1 + x^{10} \) grow more rapidly than \( y(t) \), since \( x^2 < x^{10} \) for \( 1 < x \). Thus the solutions \( x(t) \) must also blow up in finite time. This is not yet a complete argument, though, since it is only valid for \( 1 < x \); but since \( 1 \leq 1 + x^{10} \), we know that solutions beginning at any initial condition \( x_0 \) will reach \( x = 1 \) at the latest at time \( t = 1 - x_0 \); and since we reach \( x = 1 \) in finite time, we can then begin the comparison with \( y(t) \).

An alternative argument: suppose \( x(0) = x_0 \). The time taken to diverge (reach \( x = \infty \)) is given by

\[
T = \int_{x_0}^{\infty} \frac{1}{1 + x^{10}} \, dx
\]

If this is finite for all \( x_0 \), then we have finite-time blow-up. But we have

\[
T < \int_{-\infty}^{\infty} \frac{1}{1 + x^{10}} \, dx
\]

Thus an estimate of the upper bound for the blow-up time for any initial condition is

\[
> \int_{-1}^{1} dx + 2 \int_{1}^{\infty} \frac{1}{1 + x^2} \, dx
\]

The actual upper bound is

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^{10}} \, dx
\]
We plot some numerical solutions:

```plaintext
DEplot(diff(x(t),t)=1+x(t)^10,x(t),t=0..3,x=-3..5,[[x(0)=-1.1]], stepsize=0.01,linecolor=black);
```