Homework Set 3

Course Web Site: http://www.math.sfu.ca/~ralfw/math467/

1. 3.3.1

\[ \dot{n} = GnN - kn \quad (1) \]
\[ \dot{N} = -GnN - fN + p \quad (2) \]

is a model for a laser, more realistic than that treated in class; \( N \) is the number of excited atoms, \( n \) the number of laser photons, and we let the bifurcation parameter be the pump strength \( p \). We simplify this to a one-dimensional system by the quasistatic approximation:

(a) Assuming \( \dot{N} \approx 0 \), we find \( N(t) \approx p/(Gn(t) + f) \); substituting, we derive the first-order system for \( n(t) \),

\[ \dot{n} = \frac{Gpn}{Gn + f} - kn \equiv h(n). \]

(b) The fixed points (found from \( \dot{n} = 0 \)) are given by \( n^* = 0 \) and \( n^* = (Gp - kf)/kG \). The stability of the fixed point \( n^* = 0 \) is given from

\[ \frac{\partial h}{\partial n} \bigg|_{n=0} = \frac{Gpf}{(Gn + f)^2} - k \bigg|_{n=0} = \frac{Gpf}{f^2} - k = \left( \frac{Gp}{kf} - 1 \right) k. \]

Hence \( n^* = 0 \) becomes unstable when \( p > p_c = kf/G \).

(c) We expect a transcritical bifurcation, since \( n^* = 0 \) is always a fixed point, and there is no symmetry; also, the two fixed points coincide when \( p = p_c \).

Check by expanding near \( n^* = 0 \):

\[ h(n) = Gpn(f + Gn)^{-1} - kn = \frac{Gpn}{f} \left( 1 + \frac{Gn}{f} \right)^{-1} - kn \]
\[ = \frac{Gpn}{f} \left( 1 - \frac{G}{f} n + \frac{G^2}{f^2} n^2 - \ldots \right) - kn \]
\[ = \left( \frac{Gp}{f} - k \right) n - \frac{G^2 p}{f^2} n^2 + O(n^3), \]

which yields the transcritical bifurcation normal form near \( n^* = 0 \), and the bifurcation value \( p_c = kf/G \).

(d) A typical time scale for decay of \( n \) is \( 1/k \), for decay of \( N \) is \( 1/f \). We expect that the adiabatic approximation is valid if \( N \) equilibrates rapidly to changes in \( n \), that is, if \( N \) relaxes over much shorter time scales (or decays much more quickly), so we require \( 1/f \ll 1/k \), or \( f \gg k \). This can be confirmed by a careful nondimensionalization.

2. 3.4.8

\[ \dot{x} = rx - \frac{x}{1 + x^2} \equiv f(x, r) \]

(symmetric under the map \( x \mapsto -x \)).

The fixed point \( x^* = 0 \) exists for all \( r \). There are nontrivial fixed points satisfying \( r - 1/(1 + x^2) = 0 \), or \( x^* = \pm \sqrt{1/r - 1} \); these exist provided \( 1/r - 1 > 0 \), or \( 0 < r < 1 \).
We determine the stability of the fixed point at the origin by computing

\[ \left. \frac{\partial f}{\partial x} \right|_{x=0} = r - \left. \frac{1 - x^2}{(1 + x^2)^2} \right|_{x=0} = r - 1, \]

indicating that \( x^* = 0 \) is linearly stable for \( r < 1 \) and unstable for \( r > 1 \), so we expect a *subcritical pitchfork* bifurcation at \( r = 1, x = 0 \). We can verify this by expanding near the bifurcation point \( x = 0, r = 1 \), and comparing with the normal form:

\[ f(x, r) = rx - x(1 + x^2)^{-1} = (r - 1)x + x^3 + O(x^5), \]

confirming the presence of a subcritical pitchfork bifurcation at \( r = 1, x = 0 \).

![Bifurcation diagram](image)

Figure 1: Bifurcation diagram \( r = 1/(1 + x^2) \), showing the subcritical pitchfork bifurcation; solid lines indicate stable branches, dashed curves are unstable.

What happens to the (unstable) nontrivial fixed points \( x^* = \pm \sqrt{1/r - 1} \) at \( r = 0 \) (they collide with \( x^* = 0 \) at \( r = 1 \))? Note that these fixed points satisfy \( |x^*| \to \infty \) as \( r \to 0 \); and in fact they collide with the “point at infinity”:

To study the behaviour near infinity, set \( y = 1/x \), then \( |x| \to \infty \) corresponds to \( y \to 0 \). We find the differential equation for \( y \) using \( \dot{y} = -\dot{x}/x^2 \), and substituting \( x = 1/y \); after some algebra we find

\[ \dot{y} = (1 - r)y - \frac{y}{1 + y^2}; \]

expanding near \( y = 0 \) (that is, near the “point at infinity”),

\[ \dot{y} = -ry + y^3 + O(y^5). \]

Thus \( y = 0 \) is unstable for \( r < 0 \), stable for \( r > 0 \); as \( r \) decreases through 0, \( y = 0 \) undergoes a subcritical pitchfork bifurcation, so for \( r > 0 \), there are two unstable fixed points \( y^* = \pm \sqrt{r/(1 - r)} \) (that is, \( x^* = \pm \sqrt{(1 - r)/r} \)), which bifurcate from \( y = 0 \), that is, from the “point at infinity” in the original dynamical system.

3. **3.4.11**

\[ \dot{x} = rx - \sin x \]

(a) For \( r = 0 \), we have \( \dot{x} = -\sin x \), with fixed points at \( x = n\pi, n = 0, \pm 1, \pm 2, \ldots \). The fixed points with \( n \) even are stable, \( n \) odd are unstable.

(b) When \( r > 1 \), the unique fixed point is an unstable fixed point at \( x = 0 \).
Figure 2: Vector field for $r = 0$; solid circles indicate stable fixed points, open circles show unstable equilibria.

Figure 3: Plot of $rx$ and $\sin x$ for $r = 2$, showing that for $r > 1$ there is a unique fixed point.

(c) The easiest way to study this problem is to look at intersections of the functions $f(x) = rx$ and $f(x) = \sin x$, which yield the locations of the fixed points. As $r$ decreases through 1, the origin stabilizes in a subcritical pitchfork bifurcation, and two new unstable fixed points are created. As $r$ decreases further, tangencies between $rx$ and $\sin x$ occur successively at the second, third, fourth, . . . peak of $\sin x$; each of these is a saddle-node bifurcation generating a stable (for smaller $|x|$) and unstable (larger $|x|$) fixed point. Due to the symmetry $x \mapsto -x$, each bifurcation occurring for positive $x$ values is twinned to a simultaneous bifurcation at negative $x$ values. In summary, there are infinitely many pairs of saddle-node bifurcations as $r$ decreases from $\infty$ to 0.

Figure 4: Plot of $rx$ and $\sin x$ for successively decreasing values of $r$; as the straight line becomes tangent to and then intersects successive peaks of the sine function, there are infinitely many saddle-node bifurcations (in pairs).

(d) The $n$th bifurcation ($n > 1$) involves the tangency of $rx$ with the $n$th positive peak of the sine function, which is centered at $x = (2n - 3/2)\pi$ (for $x > 0$; with a simultaneous bifurcation for $x < 0$). For small $r$, or large $n$, the line $rx$ has slope near 0, and the tangency occurs near the maximum of the peak, that is, near $x_n \approx (2n - 3/2)\pi$, where $\sin x \approx 1$. Thus the $n$th bifurcation value of $r$ satisfies, for large $n$, $r_n x_n = \sin x_n \approx 1$, so $r_n \approx 1/x_n = 1/(2n - 3/2)\pi$. 
Figure 5: Plot of $rx$ and $\sin x$ for $r = r_5 = 1/(2 \cdot 5 - 3/2)\pi$, showing that this is the (approximate) 5th bifurcation value of $r$.

(e) As decreases from 0 to $-\infty$, pairs of fixed points collide in (infinitely many) saddle-node bifurcations and are annihilated, until for sufficiently negative $r$ (less than about -0.22) the only remaining fixed point is the stable fixed point at $x = 0$.

Figure 6: Plot of $rx$ and $\sin x$ for negative and decreasing values of $r$; pairs of fixed points are successively annihilated in saddle-node bifurcations.

(f) The bifurcation diagram is given by the curves $x = 0$ and $r = \sin x/x$.

Figure 7: Bifurcation diagram $r = \sin x/x$; solid lines indicate stable branches, dashed curves are unstable.

4. 3.4.12 Quadfurcation
A saddle-node bifurcation has the normal form $\dot{x} = r - x^2$: no fixed points for $r < 0$, two fixed points for $r > 0$, bifurcating from $x = 0$.
A pitchfork “trifurcation”, with $\dot{x} = rx - x^3 = x(r - x^2)$, has one branch of fixed points for $r < 0$, three branches for $r > 0$, bifurcating from $x = 0$.
One possibility to construct a “quadfurcation” is to have multiple saddle-node bifurcations occurring simultaneously: for example, for
$$\dot{x} = r - (x - 1)^2(x + 1)^2,$$
for \( r < 0 \) there are no fixed points, while for \( r > 0 \) there are four branches of fixed points; at \( r = 0 \), saddle-node bifurcations occur simultaneously at \( x = +1 \) and \( x = -1 \).

We can extend this idea to arbitrarily many branches of fixed points: for \( n = 1, 2, 3, \ldots \), let \( x_1, x_2, \ldots, x_n \) be \( n \) distinct real numbers. Then the (even-order) dynamical system

\[
\dot{x} = r - (x - x_1)^2(x - x_2)^2 \cdots (x - x_n)^2 = r - \prod_{j=1}^n (x - x_j)^2
\]

has no fixed points for \( r < 0 \), and \( 2n \) fixed points for \( r > 0 \), born out of \( n \) simultaneous saddle-node bifurcations at \( r = 0 \).

Similarly, the (odd-order) system

\[
\dot{x} = x \left[ r - (x - x_1)^2(x - x_2)^2 \cdots (x - x_n)^2 \right]
\]

has a fixed point at \( x = 0 \) for all \( r \), and \( 2n \) new fixed points created (in saddle-node bifurcations) as \( r \) increases through 0.

Here is another approach to obtaining a “quadfurcation”, and by extension, multiple branches: Consider for instance

\[
\dot{x} = -(r - x^2)(2r - x^2),
\]

which has no fixed points for \( r < 0 \), and four branches of fixed points for \( r > 0 \), at \( x = \pm \sqrt{r} \) and \( x = \pm \sqrt{2r} \); all emerge from \( x = 0 \) at \( r = 0 \). Note that the upper fixed point at \( x = +\sqrt{2r} \) is stable, as is the point \( x = -\sqrt{r} \).

We can generalize this as follows: let \( a_1, a_2, \ldots, a_n \) be \( n \) distinct positive real numbers, \( a_j > 0 \); without loss of generality, choose \( 0 < a_1 < a_2 < \cdots < a_n \). Then the dynamical system

\[
\dot{x} = (-1)^{n+1} \left( a_1 r - x^2 \right) \left( a_2 r - x^2 \right) \cdots \left( a_n r - x^2 \right)
\]

has no fixed points for \( r < 0 \), and \( 2n \) fixed points for \( r > 0 \), all created in a bifurcation at \( r = 0 \), \( x = 0 \); with the given choice of sign, the largest fixed point, at \( x = +\sqrt{a_n r} \), is stable for \( r > 0 \).

Similarly, the system

\[
\dot{x} = (-1)^{n+1} x \left( a_1 r - x^2 \right) \left( a_2 r - x^2 \right) \cdots \left( a_n r - x^2 \right)
\]

has a unique (stable) fixed point at \( x = 0 \) for \( r < 0 \), and \( 2n + 1 \) fixed points for \( r > 0 \), born in a generalized pitchfork bifurcation from \( x = 0 \), \( r = 0 \). If \( n \) is even, then \( x = 0 \) remains stable for \( r > 0 \) beyond the bifurcation.

5. **3.4.14 Subcritical Pitchfork Bifurcation**

\[
\dot{x} = rx + x^3 - x^5
\]

(a) The fixed points are \( x^* = 0 \) and the solutions of \( r + x^2 - x^4 = 0 \) (an algebraic equation in \( x^2 \)), with solutions

\[
x^* = \pm \sqrt{\frac{1}{2} \pm \sqrt{r + \frac{1}{4}}}
\]

(there are four nontrivial fixed points corresponding to four choices of sign \( \pm / \pm \)). When \( r < -1/4 \), none of the four nontrivial fixed points exists; they are all created in a saddle-node bifurcation at \( r = -1/4 \).

(b) See Figure 8, below.

(c) The (double) saddle-node bifurcation is at \( r = -1/4 \), as seen from the formula for the nontrivial fixed points in a) above; or from writing the dynamical system in the equivalent form

\[
\dot{x} = rx + x^3 - x^5 = x \left( r + \frac{1}{4} - \left( x^2 - \frac{1}{2} \right)^2 \right),
\]

from which it is clear that the fixed points are born at \( x = \pm 1/\sqrt{2} \) when \( r = 1/4 \).
Figure 8: (a)–(e): The vector field \( \dot{x} = rx - x^3 + x^5 \) and its fixed points (solid circle: stable; open circle: unstable) for (a) \( r < -1/4 \) (one stable fixed point); (b) \( r = -1/4 \) (saddle-node bifurcations: \( x = 0 \) is stable, the new fixed points at \( x = \pm 1/\sqrt{2} \) are semistable); (c) \(-1/4 < r < 0\); (d) \( r = 0 \) (subcritical pitchfork bifurcation); (e) \( r > 0 \) (one unstable and two stable fixed points). (f) Bifurcation diagram for subcritical pitchfork bifurcation, with saddle-node bifurcations at \( r = -1/4 \) and the pitchfork bifurcation at \( r = 0 \); solid curves denote stable branches, dashed curves are unstable.

6. Additional problem: Numerical Bifurcation Diagrams

The solution to (a) is given in the text and lecture notes. Following the description on the problem sheet, here is a set of Matlab commands that produces the desired plots for (b):

```matlab
% This file collects a sequence of commands to compute the bifurcation
% picture for the imperfect bifurcation problem of func3.

% Note that this is a set of commands that is obtained by some Matlab
% trial and error, and is a summary of the commands that worked... These
% commands shouldn't be run as a single M-file; in particular, at some
% points the value of b in the file func3 was changed.

% First, following (more-or-less) the instructions in the problem set,
% let's compute a nice bifurcation diagram for b = 0.1:
% x = -1:0.02:1.5;
figure(1);
cf;
plot(x, func3(x,1), 'b', [-1 1.5], [0 0], ':')
x1 = fzero('func3(x,1)', [-1 -0.5])
x2 = fzero('func3(x,1)', [-0.5 0.5])
x3 = fzero('func3(x,1)', [0.5 1.5])
% We find x1 = -0.9456, x2 = -0.1010, x3 = 1.0467
figure(2)
cf; hold on;
bifurcationplot('vfield3', x1, 1, -0.9, 4);
bifurcationplot('vfield3', x3, 1, -0.8, 3);
% Now save, print and annotate this figure
```
% Now compute similar bifurcation diagrams for \( b = 0 \) and \( b < 0 \).
% Change the 4th line of func3.m to read \( b = 0.0 \); (and save func3.m)
% We notice that we need a larger range of values of \( x \):
\[
x = -1.5:0.02:1.5;
\]
plot(x,func3(x,1),'b',[-1.5 1.5],[0 0],':');
x1 = fzero('func3(x,1)', [-1 -0.5])
x2 = fzero('func3(x,1)', [-0.5 0.5])
x3 = fzero('func3(x,1)', [0.5 1.5])
% We find \( x_1 = -1 \), \( x_2 = 0 \), \( x_3 = 1 \), which we could have written down
% immediately. However, we notice that for \( b = 0 \) this is a pitchfork
% bifurcation (which we also notice when we try to plot the bifurcation
% diagram), and the curves of fixed points cannot be integrated past the
% bifurcation point. Thus in order to obtain a branch for \( a < 0 \), we need
% to find a fixed point for negative \( a \).
plot(x,func3(x,-1),'b',[-1.5 1.5],[0 0],':');
% This plot shows that there is only a single fixed point for \( a < 0 \), at
% \( x_4 = 0 \), which we can find either analytically or numerically.
x4 = fzero('func3(x,-1)',[-1 1])

% Now plot the bifurcation diagram (the values of \( s_i \) and \( s_f \) used here were
% obtained by trial and error to get the best graphs).
figure(2); clf; hold on;
bifurcationplot('vfield3',x1,1,-1,4);
bifurcationplot('vfield3',x2,1,-3,2);
bifurcationplot('vfield3',x3,1,-1,4);
bifurcationplot('vfield3',x4,-1,-4,1.2);
% Save and print...

% Lastly, set \( b = -0.1 \) in func3.m
x = -1.5:0.02:1;
plot(x,func3(x,1),'b',[-1.5 1],[0 0],':');
x1 = fzero('func3(x,1)', [-1 -0.5])
x2 = fzero('func3(x,1)', [-0.5 0.5])
x3 = fzero('func3(x,1)', [0.5 1.5])
% We find \( x_1 = -1.0467 \), \( x_2 = 0.1010 \), \( x_3 = 0.9456 \) (symmetry with \( b=0.1 \))

figure(2); clf; hold on;
bifurcationplot('vfield3',x1,1,-0.8,3);
bifurcationplot('vfield3',x3,1,-0.9,4); % ... and save, print, ...
Figure 9: Bifurcation diagrams for $\dot{x} = b + ax - x^3$ giving the fixed points $x^*$ as functions of $a$ for (a) $b = 0.1$; (b) $b = 0$; (c) $b = -0.1$. 