Math 467-3 Dynamical Systems

Brief History

Text: Steven Strogatz "Nonlinear Dynamics and Chaos"

Newton - differential equations, planetary motion

developments in calculus, DEs, classical mechanics, celestial mechanics

Poincaré - geometric approach, chaos

Cartwright, Littlewood, von der Pol, ... nonlinear oscillators

Birkhoff, Kolmogorov, ... Hamiltonian mechanics

Lorenz - atmospheric convection - chaos, strange attractor

Ruelle, Takens turbulence

May - chaos in iterated maps, population biology

Feigenbaum universality in transition to chaos

Winfree - biological oscillations

Mandelbrot - fractals

1980s → present : Widespread interest in chaos, fractals, oscillators and applications.

- systems evolving with time

settle down - equilibrium
oscillate - cycles
more complicated

Applications: eg mechanics, fluid dynamics, electrical oscillations, chemical kinetics, population biology, ecology, epidemiology etc
-we will take our examples from these fields

Course (and text) outline: simpler to more complicated..

- 1-d: Ch 2,3,4 Flows: qualitative behaviour, fixed points
  introduction to stability and bifurcation

  Ch 10 Maps: iterative maps, chaos

- 2-d: Ch 5,6,7,8 Linear systems, phase plane analysis
  linear stability
  periodic orbits, limit cycles
  Poincaré - Bendixson theorem
  Hopf, homoclinic bifurcation
  Centre manifold reduction, Poincaré maps

- Higher dimensions:
  $\geq$ 3-d dynamical systems, Lorenz equations
  Lyapunov exponents, area-preserving maps,
  homoclinic tangles, strange attractors, ...
  as time permits

Dynamical System: - a system with a deterministic rule specifying how the system evolves with time

- **ODE** $\dot{x} = f(x)$ $\qquad \dot{x} = \frac{dx}{dt}$

  continuous time

  finite-dimensional

  mechanical systems, electrical circuits, lasers, chemical kinetics, population dynamics, ...

- **PDE** $\frac{\partial u}{\partial t} = \Phi(u)$ $\qquad \Phi$ - operator, contains spatial derivatives

  spatial degrees of freedom

  infinite-dimensional

  diffusion, wave propagation, fluid dynamics - turbulence, plasma physics, electromagnetism, quantum mechanics, ...

- **Maps** $x \mapsto F(x)$

  $x_{n+1} = F(x_n)$ difference equations

  discrete time

  discretization of continuous systems, Newton's method $g(x)=0$: $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$

  digital systems, ...

General framework for ODEs:

$$\text{1-d:} \quad \dot{x} = f(x) \qquad x \in \mathbb{R},\ f : \mathbb{R} \to \mathbb{R}$$

higher dimensions: $\dot{\vec{x}} = \vec{F}(\vec{x})$.

$\vec{x}$: state of system $\qquad \vec{x} \in M$

M : phase space

$\vec{F} : M \to M$ — need $\vec{F}$ and M to define a dynamical system

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \}$$

$$\mathbb{C}^n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{C} \}$$

Continuous-time, finite-dimensional dynamical system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots, x_n) & x_1(0) = x_{10} \\ \dot{x}_2 = f_2(x_1, x_2, \dots, x_n) & x_2(0) = x_{20} \\ \vdots & \vdots \\ \dot{x}_n = f_n(x_1, x_2, \dots, x_n) & x_n(0) = x_{n0} \end{cases}$$

more compactly:

$$\dot{\vec{x}} = \vec{F}(\vec{x}), \quad \vec{x}(0) = \vec{x}_0 \qquad \vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

n-dimensional / n^{th} order system

Example: damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Write in above form: $\quad x_1 = x \quad x_2 = \dot{x}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m} x_1 - \frac{b}{m} x_2 \end{cases} \quad \text{or} \quad \dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \vec{x}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- A higher-order ODE (or system) can always be written as a system of first-order ODEs.

$$\dot{\vec{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}}_{A} \vec{x}$$ example of a <u>linear</u> system

General linear system: $\dot{\vec{x}} = A\vec{x} + \vec{c}$

$\vec{x} = \vec{x}(t) \in \mathbb{R}^n$

$A \in \mathbb{R}^{n\times n}$ matrix

$\vec{c} \in \mathbb{R}^n$

$\vec{c} \equiv \vec{0}$: homogeneous

$\vec{c} \neq \vec{0}$: nonhomogeneous

$A, \vec{c}$ independent of t : autonomous

A and/or $\vec{c}$ depends on t : nonautonomous

Linear, homogeneous system $\dot{\vec{x}} = A\vec{x}$

- Principle of <u>**Superposition**</u> !!

- if $\vec{x}_1$, $\vec{x}_2$ are solutions, so is any linear combination

eg $\vec{y} = 2\vec{x}_1 + \vec{x}_2$ satisfies $\dot{\vec{y}} = A\vec{y}$

$$\dot{\vec{y}} = 2\dot{\vec{x}}_1 + \dot{\vec{x}}_2 = 2(A\vec{x}_1) + (A\vec{x}_2)$$
$$= A(2\vec{x}_1 + \vec{x}_2) = A\vec{y}$$

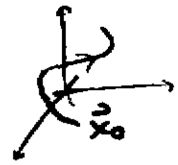
- the solution is the "sum of its parts"

Similar considerations hold for difference equations

Autonomous system: $\dot{\vec{x}} = \vec{F}(\vec{x})$ $\qquad \vec{x} \in M$, $\vec{F} : M \to M$

$\vec{x}(t)$: <u>trajectory</u> in phase space

Autonomous: evolution depends only on the state of the system



Non-autonomous: $\dot{\vec{x}} = \vec{F}(\vec{x}, t)$
explicit time-dependence

<u>But</u>: we can always consider a n-dimensional non-autonomous (time-dependent) system as a (n+1)-dimensional autonomous system

trick: introduce $x_{n+1} \equiv t$

$$\begin{cases} \dot{x}_1 = F_1(x_1, \ldots, x_n, x_{n+1}) \\ \vdots \\ \dot{x}_n = F_n(x_1, \ldots, x_n, x_{n+1}) \\ \dot{x}_{n+1} = 1 \end{cases}$$

eg forced damped oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos t$$

2-d linear non-autonomous
3-d nonlinear autonomous

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{F}{m}\cos x_3 \\ \dot{x}_3 = 1 \end{cases}$$

Review: solution of 1-d linear, autonomous ODE

$$\dot{x} = ax$$

$a > 0$

$a < 0$

Method 1: Separation of variables

$$\frac{dx}{dt} = ax \Rightarrow \int \frac{dx}{x} = a \int dt \Rightarrow \ln|x| = at + c$$

$(x \neq 0)$

$$\Rightarrow x = Ae^{at}, \quad A \in \mathbb{R}$$

$$x(0) = x_0 \Rightarrow \underline{x(t) = x_0 e^{at}}$$

– works for any first-order autonomous

$$\dot{x} = f(x) \Rightarrow \int \frac{dx}{f(x)} = t + c$$

Careful! consider critical points separately! (where $f(c) = 0$).

Method 2: Integrating factor

$$\dot{x} - ax = 0 \qquad \rho(t) = e^{-at}$$

$$e^{-at}\dot{x} - ae^{-at}x = 0 \Rightarrow \frac{d}{dt}(e^{-at}x) = 0$$

$$\Rightarrow e^{-at}x - e^0 x_0 = 0 \Rightarrow x(t) = x_0 e^{at}$$

– works for

$$\dot{x} - a(t)\,x = b(t)$$

Integrating factor $\rho(t) = e^{-\int_0^t a(s)ds}$

$$\Rightarrow \ldots \Rightarrow x(t) = e^{\int_0^t a(s)ds}\left[x_0 + \int_0^t b(s)\rho(s)\,ds\right]$$

Method 3: $\dot{x} - ax = 0$ Ansatz $x(t) = e^{\lambda t}$

Substitute: $\lambda e^{\lambda t} - ae^{\lambda t} = 0 \Rightarrow \underline{\lambda = a}$

– Works for higher-order equations (constant coefficient)

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

Linear autonomous systems $\dot{\vec{x}} = A\vec{x}$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Ansatz: assume $\vec{x}(t) = e^{\lambda t}\vec{v}$ $\qquad \vec{v} \neq \vec{0}$

$$\Rightarrow \quad \dot{\vec{x}} = \lambda e^{\lambda t}\vec{v}$$

Substitute:

$$\lambda \cancel{e^{\lambda t}}\vec{v} = A\cancel{e^{\lambda t}}\vec{v} \quad \Rightarrow \quad A\vec{v} = \lambda\vec{v}$$

$\vec{v}$ is an eigenvector of A, λ the corr. eigenvalue

$$(A - \lambda I)\vec{v} = \vec{0} \quad \Rightarrow \quad \det(A - \lambda I) = 0$$

characteristic equation

If A has n independent eigenvectors (diagonalizable) then general solution is given by superposition of particular solutions:

$$\vec{x}(t) = \sum_{i=1}^{n} a_i \vec{v}_i e^{\lambda_i t}$$

"normal modes"

Solutions are oscillatory $(\operatorname{Im}\lambda \neq 0)$

or relaxational $(\operatorname{Im}\lambda = 0)$

Nonlinear systems

- are hard
  - no principle of superposition

    normal modes
    Laplace transform } fail
    Fourier analysis
  - exact analytical solutions are rare
  - must use

    numerical methods
    perturbation, asymptotic methods
    qualitative (geometric) approach

- are ubiquitous
  - most systems are nonlinear;
    linearity is usually a simplifying
    approximation

- are interesting
  - eg finite-time singularities,
    deterministic chaos
    entrainment + synchronization
    ...

Deterministic Chaos

- **sensitive dependence on initial conditions**

small perturbations in initial values can cause
completely different long-term behaviour
- loss of predictability "butterfly effect"
- complex behaviour, looks random
(but still some structure)