Brief History

Newton - differential equations, planetary motion development in calculus, DEs, classical mechanics, celestial mechanics

Poincaré - geometric approach, chaos

Carwright, Littlewood, nonlinear oscillators
von der Pol, ...

Birkhoff, Kolmogorov, ... Hamiltonian mechanics

Lorenz - atmospheric convection - chaos, strange attractor

Ruelle, Takens - turbulence

May - chaos in iterated maps, population biology

Feigenbaum - universality in transition to chaos

Winfree - biological oscillations

Mandelbrot - fractals

1980s to present - Widespread interest in chaos, fractals, oscillators and applications.
systems evolving with time
settle down - equilibrium
oscillate - cycles
more complicated

Applications: eg mechanics, fluid dynamics,
electrical oscillations, chemical kinetics,
population biology, ecology, epidemiology etc
- we will take our examples from these fields

Course (and text) outline: simpler to more complicated...

• 1-d: Flows; qualitative behaviour, fixed points
  introduction to stability and bifurcation
  ch 10 Maps: iterative maps, chaos

• 2-d: ch 5, 6, 7, 8
  linear systems, phase plane analysis
  linear stability
  periodic orbits, limit cycles
  Poincaré - Bendixson theorem
  Hopf, homoclinic bifurcation
  Centre manifold reduction, Poincaré maps

• Higher dimensions:
  2, 3-d dynamical systems, Lorenz equations
  Lyapunov exponents, area-preserving maps,
  homoclinic tangles, strange attractors,...
  as time permits
Dynamical System: a system with a deterministic rule specifying how the system evolves with time

- **ODE** \[ \dot{x} = f(x) \] continuous time finite-dimensional mechanical systems, electrical circuits, lasers, chemical kinetics, population dynamics,...

- **PDE** \[ \frac{\partial x}{\partial t} = \Phi(u) \] \[ \Phi \] - operator, contains spatial derivatives spatial degrees of freedom infinite-dimensional diffusion, wave propagation, fluid dynamics - turbulence, plasma physics, electromagnetism, quantum mechanics,...

- **Maps** \[ x \rightarrow F(x) \] difference equations discretization of continuous systems, Newton's method \[ g(x) = 0 : x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \] digital systems,...
General framework for ODEs:

- ODE: \( x = f(x) \), \( x \in \mathbb{R} \), \( f : \mathbb{R} \to \mathbb{R} \)

Higher dimensions: \( \dot{x} = \mathbf{F}(\mathbf{x}) \).

\( x \): state of system \( x \in \mathbb{R} \)

\( M \): phase space

\( \mathbf{F} : M \to M \) - need \( \mathbf{F} \) and \( M \) to define a dynamical system

\( \mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \} \)

\( \mathbb{C}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{C} \} \)

Continuous-time, finite-dimensional dynamical system

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \quad x_1(0) = x_{10} \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n) \quad x_2(0) = x_{20} \\
& \vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) \quad x_n(0) = x_{n0}
\end{align*}
\]

More compactly:

\( \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \), \( \mathbf{x}(0) = \mathbf{x}_0 \)

\( \mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n \)

n-dimensional \( n^{th} \) order system

Example: damped harmonic oscillator

\( m \ddot{x} + b \dot{x} + kx = 0 \)

Write in above form:

\( x_1 = x \quad x_2 = \dot{x} \)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m} x_1 - \frac{b}{m} x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} 0 & 1 \\ \frac{k}{m} & \frac{b}{m} \end{pmatrix} \ddot{\mathbf{x}}, \quad \dot{\mathbf{x}} = (x_2)
\end{align*}
\]

- A higher-order ODE (or system) can always be written as a system of first-order ODEs.
\[
\dot{x} = \begin{pmatrix}
0 & 1 \\
-k & -\frac{b}{m}
\end{pmatrix} x
\]

Example of a linear system

General linear system: \[ \dot{x} = A x + \bar{z} \]

\[ x = \bar{x}(t) \in \mathbb{R}^n \]

\[ A \in \mathbb{R}^{n \times n} \quad \text{matrix} \]

\[ \bar{z} \in \mathbb{R}^n \]

\[ \bar{z} = 0 : \text{homogeneous} \]

\[ \bar{z} \neq 0 : \text{nonhomogeneous} \]

A, \bar{z} independent of \( t \): autonomous

A and/or \( \bar{z} \) depends on \( t \): nonautonomous

Linear, homogeneous system \[ \dot{x} = A x \]

- Principle of Superposition !!!

- If \( x_1, x_2 \) are solutions, so is any linear combination

\[ \bar{y} = 2x_1 + x_2 \]

satisfies \[ \dot{y} = Ay \]

\[ \dot{y} = 2\dot{x}_1 + \dot{x}_2 = 2(Ax_1) + (Ax_2) \]

\[ = A(2x_1 + x_2) = Ay \]

- The solution is the "sum of its parts"

Similar considerations hold for difference equations
Autonomous system: \( \dot{x} = F(x) \quad x \in \mathbb{R}^n \quad F: \mathbb{R}^n \to \mathbb{R}^n \)

\( \dot{x}(t) \): trajectory in phase space

Autonomous: evolution depends only on the state of the system

Non-autonomous: \( \dot{x} = F(x, t) \)

explicit time-dependence

But: we can always consider a \( n \)-dimensional non-autonomous (time-dependent) system as a \( (n+1) \)-dimensional autonomous system

trick: introduce \( x_{n+1} = t \)

\[
\begin{align*}
\dot{x}_1 &= F_1(x_1, \ldots, x_n, x_{n+1}) \\
\vdots \\
\dot{x}_n &= F_n(x_1, \ldots, x_n, x_{n+1}) \\
\dot{x}_{n+1} &= 1
\end{align*}
\]

eg forced damped oscillator

\( m \ddot{x} + b \dot{x} + k x = f \cos t \)

2-d linear non-autonomous

3-d nonlinear autonomous

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m} x_1 - \frac{b}{m} x_2 \\
\dot{x}_3 &= -\frac{k}{m} \cos x_3 \\
\end{align*}
\]

\( \dot{x}_3 = 1 \)
Review: solution of 1-d linear, autonomous ODE
\[ \dot{x} = ax \quad a > 0 \]

Method 1: Separation of variables
\[ \frac{dx}{dt} = ax \Rightarrow \int \frac{dx}{x} = \int a \, dt \Rightarrow \ln |x| = at + c \]
\[ x(t) = x_0 e^{at} \quad x_0 \in \mathbb{R} \]
\[ x(0) = x_0 \Rightarrow x(t) = x_0 e^{at} \]
- works for any first-order autonomous

\[ x = f(x) \Rightarrow \int \frac{dx}{f(x)} = t + c \]

Careful! consider critical points separately!
(where \( f(c) = 0 \)).

Method 2: Integrating factor
\[ \dot{x} = -ax = 0 \quad \Rightarrow \quad x(t) = e^{-at} \]
\[ e^{-at} \dot{x} - ae^{-at} x = 0 \Rightarrow \frac{d}{dt}(e^{-at} x) = 0 \]
\[ e^{-at} x = e^{0} x_0 = 0 \Rightarrow x(t) = x_0 e^{at} \]
- works for

\[ \dot{x} = a(t) x = b(t) \]

Integrating factor \( \varphi(t) = e^{\int_{0}^{t} a(s) \, ds} \)
\[ \Rightarrow \quad x(t) = e^{\int_{0}^{t} a(s) \, ds} \left[ x_0 + \int_{0}^{t} b(s) \varphi(s) \, ds \right] \]

Method 3: \( \dot{x} = -ax = 0 \quad \text{Ansatz:} \quad x(t) = e^{\lambda t} \)

Substitute:
\[ \lambda e^{\lambda t} - ae^{\lambda t} = 0 \Rightarrow \lambda = a \]
- works for higher-order equations (constant coefficient)
\[ a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_2 \dot{x} + a_1 x + a_0 x = 0 \]
Linear autonomous systems \( \dot{\vec{x}} = A \vec{x} \)

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{pmatrix}
=
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

Ansatz: assume \( \dot{\vec{x}}(t) = e^{\lambda t} \vec{\tilde{v}} \) \( t \neq 0 \)

\[\Rightarrow \dot{\vec{x}} = \lambda e^{\lambda t} \vec{\tilde{v}}\]

Substitute:
\[\lambda e^{\lambda t} \vec{\tilde{v}} = A e^{\lambda t} \vec{\tilde{v}} \Rightarrow A \vec{\tilde{v}} = \lambda \vec{\tilde{v}}\]

\( \vec{\tilde{v}} \) is an eigenvector of \( A \), \( \lambda \) the corr. eigenvalue

\[
(A - \lambda I) \vec{\tilde{v}} = \vec{0} \Rightarrow \det(A - \lambda I) = 0
\]

characteristic equation

If \( A \) has \( n \) independent eigenvectors (diagonalizable) then general solution is given by superposition of particular solutions:

\[\vec{x}(t) = \sum_{i=1}^{n} a_i \vec{\tilde{v}}_i e^{\lambda_i t} \] "normal modes"

Solutions are oscillatory \( (\text{Im } \lambda \neq 0) \)
or relaxation \( (\text{Im } \lambda = 0) \)
Nonlinear systems

- are hard
  - no principle of superposition
  - normal modes (Laplace transform) foil Fourier analysis
  - exact analytical solutions are rare
  - must use numerical methods
    - perturbation, asymptotic methods
    - qualitative (geometric) approach

- are ubiquitous
  - most systems are nonlinear
  - linearity is usually a simplifying approximation

- are interesting
  - e.g. finite time singularities
  - deterministic chaos
  - entrainment + synchronization

Deterministic Chaos

- sensitive dependence on initial conditions
- small perturbations in initial values can cause completely different long-term behaviour
- loss of predictability "butterfly effect"
- complex behaviour looks random (but still some structure)